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Critical Curve for *p*-*q* Systems of Nonlinear Wave Equations in Three Space Dimensions

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The existence of the critical curve for p-q systems for nonlinear wave equations was already established by D. Del Santo, V. Georgiev, and E. Mitidieri [1997, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, *in* "Geometric Optics and Related Topics" (F. Colombini and N. Lerner, Eds.), Progress in Nonlinear Differential Equations and Their Applications, Vol. 32, pp. 117–139, Birkhäuser, Basel] except for the critical case. Our main purpose is to prove a blow-up theorem for which the nonlinearity (p, q) is just on the critical curve in three space dimensions. Moreover, the lower and upper bounds of the lifespan of solutions are precisely estimated, including the sub-critical case.

Key Words: nonlinear wave equations; three space dimension; lifespan.

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1. INTRODUCTION

We are concerned with the Cauchy problem for p-q systems of nonlinear wave equations

$$\begin{aligned} \Box u &= |v|^p, \\ \Box v &= |u|^q, \end{aligned} \qquad \text{in} \quad \mathbf{R}^n \times [0, \infty), \tag{1.1}$$

where $\Box = \partial^2/\partial_t^2 - \sum_{j=1}^n \partial^2/\partial x_j^2$ is a usual d'Alembertian in $\mathbf{R}^{n+1}$ and p, $q > 1$. The initial data take the form

$$\begin{aligned} u(x, 0) &= \varepsilon f_1(x), \qquad (\partial u/\partial t)(x, 0) = \varepsilon g_1(x), \\ v(x, 0) &= \varepsilon f_2(x), \qquad (\partial v/\partial t)(x, 0) = \varepsilon g_2(x), \end{aligned} \tag{1.2}$$

where f_i, g_i $(i = 1, 2)$ are smooth functions of compact support and ε is a small positive parameter which measures the smallness of the amplitude of solutions.

The problem (1.1) sometimes arises as an analogy of the Lane–Emden system and its associated parabolic version in which $\Box$ in (1.1) is replaced by $-\Delta$ or $\partial_t - \Delta$. See [2] for details and further references.

Recently, DelSanto *et al.* [2] proved in any space dimension $n \geqslant 2$ that there exists a critical curve in the (p, q)-plane which divides the plane into two pieces. One is a range where we can show the global-in-time existence of a small amplitude solution. Another is a range where we can give an example of the nonexistence of a global-in-time solution. We note that the critical curve is determined by a cubic relation between p and q, and has a cusp at $p = q$.

More precisely, defining

$$F(p, q) \equiv \max\left\{\frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1}\right\} - \frac{n-1}{2}, \tag{1.3}$$

they proved the following fact: If $F(p, q) < 0$, the system (1.1) with any data (1.2) admits a unique global solution provided ε is sufficiently small. Note that, in general, the solution must be weak whenever p or q is less than 2 because of the regularity of the nonlinearities. For this reason, the classical solution can be obtained only in the case $n = 2$, 3, or $n = 4$ at the cusp only. Conversely, if $F(p, q) > 0$, (1.1) with some positive data (1.2) has no global solution. The critical case $F(p, q) = 0$ was investigated by D. Del Santo & E. Mitidieri [3] for $n = 3$ in which the nonexistence of global solutions for some positive data was proved.

Our aim in this article is to clarify the lifespan (the maximal existence time) of the solution in three space dimensions without any positivity of data by proving the local-in-time existence and the nonexistence in the long time of solutions. Here, we restrict our attention to the classical sense so that the lifespan $T(c)$ is defined by

$$T(\varepsilon) = \sup\{T \in (0, \infty] : \text{There exists a unique solution } (u, v) \\ \in \{C^2(\mathbf{R}^n \times [0, T])\}^2 \text{ of (1.1) with any fixed data (1.2)}\}. \tag{1.4}$$

By virtue of the well-known uniqueness theorem, one has $\lim_{\varepsilon \to 0} T(\varepsilon) = \infty$. For example, see Appendix 1 in F. John [9]. We will prove the following theorem.

THEOREM 1. *Let $n = 3$ and $p, q \geqslant 2$. Suppose that both $f_i \in C_0^4(\mathbf{R}^3)$ and $g_i \in C_0^3(\mathbf{R}^3)$ do not identically vanish for each $i = 1, 2$. Then there exists a positive constant ε_0 such that, for any ε with $0 < \varepsilon \leqslant \varepsilon_0$, the lifespan $T(\varepsilon)$ of the classical solution (u, v) of* (1.1), (1.2) *satisfies*

$$T(\varepsilon) = \infty \tag{1.5}$$

provided $F(p, q) < 0$;

$$\exp(c\varepsilon^{-\min\{p(pq-1),\, q(pq-1)\}}) \leqslant T(\varepsilon) \leqslant \exp(C\varepsilon^{-\min\{p(pq-1),\, q(pq-1)\}}) \tag{1.6}$$

provided $F(p, q) = 0$ with $p \neq q$;

$$\exp(c\varepsilon^{-p(p-1)}) \leqslant T(\varepsilon) \leqslant \exp(C\varepsilon^{-p(p-1)}) \tag{1.7}$$

provided $F(p, q) = 0$ with $p = q$; and

$$c\varepsilon^{-F(p,q)^{-1}} \leqslant T(\varepsilon) \leqslant C\varepsilon^{-F(p,q)^{-1}} \tag{1.8}$$

provided $F(p, q) > 0$, where c and C are positive constants independent of ε.

Remark 1.1. The restriction $2 < p, q \leqslant 3$ in three space dimensions of the global existence theorem of [2] was relaxed by D. Del Santo [1]. Actually, he used a weighted L^∞ estimate originally introduced by F. John [8] as conjectured in Remark 1.1 of [2]. See also our proof.

Remark 1.2. In the blow-up part of Theorem 1, there is no requirement of the positivity of the initial data (cf., Theorem 3 in [2]). D. Del Santo [1] also proved the sub-critical blow-up without any positivity on data. He employed some technique by F. John [9], but it cannot be applicable to estimating the lifespan. By making use of the local existence, we will succeed in removing the positivity. Such an argument can be found in F. John [8] in which the lifespan is estimated for a single equation with a quadratic nonlinearity.

Remark 1.3. At the cusp (p, p) on the critical curve $F(p, p)=0$, p must be a number $p_0(3)=1+\sqrt{2}$ which is the critical power of the single case. See (1.14) below. Moreover, the lifespan at the cusp coincides with the one for the single case. See also (1.18) below.

Remark 1.4. For $1<p, q<2$, we cannot expect any existence of the classical solutions from the lack of the differentiability of the nonlinearity. But we may obtain the same lifespan of the C^1-solution of the integral equation associated to (1.1), (1.2) which will appear in the next section.

As in [2], it is interesting to compare the result of the p-q system with that one of the single equation

$$\begin{aligned} \Box u = |u|^p \qquad & \text{in} \quad \mathbf{R}^n \times [0, \infty) \\ u(x, 0) = \varepsilon f(x), \qquad & (\partial u/\partial t)(x, 0) = \varepsilon g(x). \end{aligned} \tag{1.9}$$

It is well known, as Strauss' conjecture [17], that the lifespan $T(\varepsilon)$ of a solution of (1.9) satisfies $T(\varepsilon)=\infty$ for small ε if

$$p > p_0(n), \tag{1.10}$$

where $p_0(n)$ is a positive root of the quadratic equation

$$\gamma(p, n) \equiv 2+(n+1)\,p-(n-1)\,p^2=0, \tag{1.11}$$

and $T(\varepsilon)<\infty$ for some special data with a positivity if

$$1<p\leqslant p_0(n) \tag{1.12}$$

which can be rewritten as

$$\frac{1+p^{-1}}{p-1} > \frac{n-1}{2} \qquad \text{and} \qquad p>1. \tag{1.13}$$

In this sense, $p_0(n)$ is a critical value of (1.9). One can find that

$$p_0(n)=\frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}, \qquad n \geqslant 2. \tag{1.14}$$

We note that $p_0(n)$ is monotonously decreasing in n and $p_0(4)=2$. Therefore, we have to consider the weaker solution rather than C^2 if p is in the neighborhood of $p_0(n)$ in higher dimensions $n \geqslant 4$.

This conjecture was verified by F. John [8] for $n=3$ and by R. T. Glassey [5, 6] for $n=2$ except for the critical case. The critical case was proved by

J. Schaeffer [15] for $n=2$, 3. The blow-up part in higher dimensions was verified by T. C. Sideris [16] except for the critical case. For the global existence part there were many partial results. A complete result was given by V. Georgiev, H. Lindblad and C. Sogge [4] in which we can find references on history. The open problem is the case $p=p_0(n)$ for $n \geqslant 4$.

As for the order of $T(\varepsilon)$, we have a few results. H. Lindblad [14] for $n=3$ and Zhou Yi [20] for $n=2$ proved that

$$\lim_{\varepsilon \to 0} \varepsilon^{2p(p-1)/\gamma(p,n)} T(\varepsilon) > 0 \qquad \text{exists for} \quad 4-n<p<p_0(n). \tag{1.15}$$

For $n=2$, $p=2$, H. Lindblad [14] proved that

$$\begin{aligned} &\lim_{\varepsilon \to 0} a^{-1}(\varepsilon)\, T(\varepsilon) > 0 \qquad \text{exists if} \quad \int_{\mathbf{R}^2} g(x)\, dx \neq 0 \\ &\lim_{\varepsilon \to 0} \varepsilon T(\varepsilon) > 0 \qquad \text{exists if} \quad \int_{\mathbf{R}^2} g(x)\, dx = 0, \end{aligned} \tag{1.16}$$

where $a=a(\varepsilon)$ satisfies

$$\varepsilon^2 a^2 \log(1+a) = 1. \tag{1.17}$$

Remark 1.5. Making use of H. Lindblad's methods, we may have a limit of the lifespan in the sub-critical case of Theorem 1. But this is another story.

Zhou Yi [19, 20] proved that there exist positive constants c, C independent of ε such that

$$\exp(c\varepsilon^{-p(p-1)}) \leqslant T(\varepsilon) \leqslant \exp(C\varepsilon^{-p(p-1)}) \qquad \text{for} \quad p=p_0(n), \quad n=2, 3. \tag{1.18}$$

By making use of the L^2-framework, Li and Zhou [13] showed that, in the case $n=4$, there exists a positive constant c independent of ε such that

$$T(\varepsilon) \geqslant \exp(c\varepsilon^{-2}) \qquad \text{for} \quad p=p_0(4)=2. \tag{1.19}$$

Remark 1.6. The proof of the blow-up result for a single equation (1.9) with a sub-critical power is essentially due to T. Kato's blow-up theorem [10] for 2nd order ordinary differential inequality. Such an inequality can be applicable to the sub-critical case of our system (1.1) by iteration argument; see [2]. The critical case for a single equation is due to Zhou Yi's blow-up theorem for 2nd order ordinary differential equations. We note that his theorem is not directly applicable to our system because we have to make a comparison argument with a system of 2nd order ordinary differential equations which is difficult to solve. Our success with the blow-up result on the critical curve is due to a logarithmic term in the iteration argument which is obtained with our new slicing method.

This article is organized as follows. In Section 2, the representation formula of the solution is clarified together with a support property of the finite propagation speed of the wave and with a decay estimate for a solution of free wave equations. In Section 3, we calculate the lower bound of the lifespan by making a weighted L^∞ *a priori* estimate which is devided into two parts up to places. One is near the light cone in which we have the long time or global existence. This will be proved in Section 4. Another is inside of the forward cone in which we mainly have the local existence and this part determines the lower bound of the lifespan. This will be proved in Section 5. In Section 6 we get the upper bound of the lifespan in the critical case by virtue of slicing in the iteration of pointwise estimates of the spherical mean of solutions. In Section 7 we also have the upper bound of the lifespan in the sub-critical case by the usual John's iteration argument.

We finally remark that Theorem 1 is also valid for $n=2$. The proof will appear in our forthcoming paper. The difficulty will lie in the proof for the lower bound of the lifespan because the strong Huygens Principle is never available in two space dimensions.

After this work was completed, we were informed of a result of H. Kubo and M. Ohta [11]. They have proved the blow-up part of Theorem 1 for $n=2, 3$, in which data must have positivity, by a comparison argument with a system of integral equations.

2. PRELIMINARIES

We shall start with the well-known integral representation formula of the solution. Namely, solutions (u, v) of (1.1), (1.2) have to satisfy the integral equations

$$\begin{aligned} u(x,t) &= \varepsilon u^0(x,t) + L(|v|^p)(x,t), \\ v(x,t) &= \varepsilon v^0(x,t) + L(|u|^q)(x,t), \end{aligned} \tag{2.1}$$

where u^0 and v^0 satisfy $\Box u^0 = \Box v^0 = 0$ with the same initial data for u/ε and v/ε respectively. Moreover $L(w)$ satisfies

$$\begin{cases} \Box L(w) = w & \text{in} \quad \mathbf{R}^n \times [0, \infty), \\ L(w)(x,0) = (\partial L(w)/\partial T)(x,0) = 0, \end{cases} \tag{2.2}$$

where $w = |v|^p$ or $|u|^q$.

In the case $n=3$, u^0, v^0, and $L(w)$ can be expressed by

$$\begin{aligned} u^0(x,t) &= \frac{\partial}{\partial t} R(f_1\,|\,x,t) + R(g_1\,|\,x,t), \\ v^0(x,t) &= \frac{\partial}{\partial t} R(f_2\,|\,x,t) + R(g_2\,|\,x,t), \end{aligned} \tag{2.3}$$

and

$$L(w)(x,t) = \int_0^t R(w(\cdot,\tau)\,|\,x, t-\tau)\,d\tau, \tag{2.4}$$

where R is defined by

$$R(g|x,t) = \frac{t}{4\pi}\int_{|\omega|=1} g(x+t\omega)\,dS_\omega. \tag{2.5}$$

In making a weighted L^∞ estimate, we naturally use the radially symmetricity of the integrand in the representation formula.

LEMMA 2.1. *Denote a spherical mean of $h(x)\in C(\mathbf{R}^3)$ at the origin with radius $r=|x|$ by*

$$\bar{h}(r) = \frac{1}{4\pi}\int_{|\omega|=1} h(r\omega)\,dS_\omega. \tag{2.6}$$

Then the identity

$$\frac{1}{4\pi}\int_{|\eta|=1} dS_\eta \frac{1}{4\pi}\int_{|\omega|=1} h(r\eta+t\omega)\,dS_\omega = \frac{1}{2rt}\int_{|t-r|}^{t+r} \lambda\bar{h}(\lambda)\,d\lambda \tag{2.7}$$

is valid. Moreover, if h is a radially symmetric function of $r=|x|$, then, for all $(r,t)\in[0,\infty)^2$,

$$\int_{|\omega|=1} h(|x+t\omega|)\,dS_\omega = \frac{2\pi}{rt}\int_{|t-r|}^{t+r}\lambda h(\lambda)\,d\lambda. \tag{2.8}$$

Proof. This is a fundamental identity for iterated spherical means. The proof can be found in F. John's [7]. We shall omit it.

Next, we shall state the dependence domain of the solution.

LEMMA 2.2. *Assume that*

$$\operatorname{supp}\{f_i(x), g_i(x)\} \subset \{|x| \leqslant k\}, \tag{2.9}$$

where $k>0$ *and* $i=1, 2$. *Then the classical solutions* (u, v) *of* (1.1), (1.2) *have to satisfy the support property*

$$\operatorname{supp}\{u(x,t), v(x,t)\} \subset \{|x| \leqslant t+k\}. \tag{2.10}$$

Proof. This well-known fact can be found in, for instance, Appendix 1 of F. John's [8]. We shall omit the proof.

In the odd-dimensional case, the so-called strong Huygens principle is valid. Namely, we have

LEMMA 2.3. *Let* $n=3$. *Assume a support property* (2.9). *Then,*

$$u^0(x,t) \equiv v^0(x,t) \equiv 0 \qquad \text{for} \quad t-|x| \geqslant k. \tag{2.11}$$

Proof. This lemma immediately follows from the representation formula of u^0 and v^0 in which one can check it by a simple inequality $|x+t\omega| \geqslant |t-|x||$.

We also make use of the following decay estimate of solutions of free wave equations.

LEMMA 2.4. *Let* $n=3$. *Assume a support property* (2.9). *Then, under the same assumption as that of Theorem* 1 *on the initial data, there exists a positive constant* C *depending only on* $\nabla_x^\alpha f_i(|\alpha| \leqslant 4)$, $\nabla_x^\beta g_i(|\beta| \leqslant 3)$, $i=1, 2$, *such that*

$$\sum_{|\alpha| \leqslant 2} |\nabla_x^\alpha u^0(x,t)|, \sum_{|\alpha| \leqslant 2} |\nabla_x^\alpha v^0(x,t)| \leqslant \frac{C}{t+|x|+2k} \tag{2.12}$$

for $-k \leqslant t-|x| \leqslant k$, $t \geqslant 0$.

Proof. The original version of this lemma can be found in F. John's [8]. For the sake of completeness, we shall give a proof. In view of the representation formula, it is sufficient to show the lemma only for u^0. For $0 \leqslant t \leqslant k$, it follows from the well-known representation

$$u^0(x,t) = \frac{1}{4\pi} \int_{|\omega|=1} \{f_1 + t\omega \cdot \nabla f_1 + t g_1\}(x+t\omega)\, dS_\omega \tag{2.13}$$

that

$$|u^0(x,t)| \leqslant \|f_1\|_{L^\infty(\mathbf{R}^3)} + k\, \|\nabla f_1\|_{L^\infty(\mathbf{R}^3)} + k\, \|g_1\|_{L^\infty(\mathbf{R}^3)}. \tag{2.14}$$

On the other hand, for $t \geqslant k$, the divergence theorem yields

$$u^0(x,t) = \frac{1}{4\pi} \int_{|\omega|=1} \omega \cdot \omega \{ f_1 + t\omega \cdot \nabla f_1 + tg_1 \}(x + t\omega)\, dS_\omega$$
$$= \frac{1}{4\pi} \int_{|\xi| \leqslant 1} \operatorname{div}_\xi \{ \xi (f_1 + t\xi \cdot \nabla f_1 + tg_1)(x + t\xi) \}\, d\xi. \tag{2.15}$$

Introducing a new variable by $y = x + t\xi$, we have

$$u^0(x,t) = \frac{1}{4\pi} \int_{|y-x| \leqslant t} t \operatorname{div} \left\{ \frac{y-x}{t} (f_1 + (y-x) \cdot \nabla f_1 + tg_1)(y) \right\} \frac{dy}{t^3}, \tag{2.16}$$

which gives us the desired estimate

$$|u^0(x,t)| \leqslant \frac{C}{t} \left(\sum_{|\alpha| \leqslant 2} \| \nabla_x^\alpha f_1 \|_{L^1(\mathbf{R}^3)} + \sum_{|\beta| \leqslant 1} \| \nabla_x^\beta g_1 \|_{L^1(\mathbf{R}^3)} \right), \tag{2.17}$$

where C is a numerical constant. Therefore the lemma is proved by the simple inequality $t \geqslant (t + |x| + 2k)/5$.

3. LOWER BOUND OF THE LIFESPAN

The lower bound of the lifespan is estimated by proving the following proposition.

PROPOSITION 3.1. *Let $n = 3$. Under the same assumptions as Theorem 1, there exists a positive constant ε_0 such that (2.1) admits a unique solution $(u, v) \in \{C^2(\mathbf{R}^3 \times [0, T])\}^2$, as far as T satisfies*

$$T \leqslant \begin{cases} \infty & \text{if } F(p,q) < 0, \\ \exp(c\varepsilon^{-\min\{p(pq-1),\, q(pq-1)\}}) & \text{if } F(p,q) = 0 \text{ with } p \neq q, \\ \exp(c\varepsilon^{-p(p-1)}) & \text{if } F(p,q) = 0 \text{ with } p = q, \\ c\varepsilon^{-F(p,q)^{-1}} & \text{if } F(p,q) > 0, \end{cases} \tag{3.1}$$

for $0 < \varepsilon \leqslant \varepsilon_0$ and some positive constant c independent of ε.

We will solve (2.1) by the classical iteration method in a suitable function space. Here and hereafter, it is sufficient to consider the case $p \leqslant q$ because, due to the symmetricity of the equation, nothing new will come from switching p and q with each other. Let us define sequences of functions $\{u_m\}$, $\{v_m\}$ by

$$\begin{aligned} u_m &= u_0 + L(|v_{m-1}|^p), \\ v_m &= v_0 + L(|u_{m-1}|^q), \end{aligned} \qquad \text{for} \quad m \geqslant 1 \quad \text{and} \quad \begin{aligned} u_0 &= \varepsilon u^0, \\ v_0 &= \varepsilon v^0. \end{aligned} \tag{3.2}$$

In order to solve this, we shall follow F. John [8]. Denote a weighted L^∞-norm of u by

$$\|u\|_j = \sup_{(x,\, t) \in \mathbf{R}^3 \times [0,\, T]} \{ w_j(|x|, t)\, |u(x, t)| \} \qquad (j = 1, 2). \tag{3.3}$$

with the weight function

$$w_1(r, t) = \begin{cases} \dfrac{t+r+2k}{k} \left(\dfrac{t-r+2k}{k} \right)^{p-2} & \text{when} \quad p > 2, \\[2ex] \dfrac{t+r+2k}{k} \left(\log 4\, \dfrac{t+r+2k}{|t-r|+2k} \right)^{-1} & \text{when} \quad p = 2, \quad q > 2, \\[2ex] \dfrac{t+r+2k}{k} \chi_1 + \dfrac{r}{k} \left(\log \dfrac{t+r+2k}{t-r+2k} \right)^{-1} \chi_2 & \text{when} \quad p = q = 2 \end{cases} \tag{3.4}$$

and

$$w_2(r, t) = \begin{cases} \dfrac{t+r+2k}{k} \chi_1 + \dfrac{r}{k} \left(\log \dfrac{t+r+2k}{t-r+2k} \right)^{-1} \chi_2 & \text{when } p = q = 2, \\[2ex] \dfrac{t+r+2k}{k} \left(\dfrac{t-r+2k}{k} \right)^{1/p} \left(\log \dfrac{t-r+3k}{k} \right)^{\nu} & \text{when } F(p, q) = 0 \text{ with } p \neq q, \\[2ex] \dfrac{t+r+2k}{k} \left(\dfrac{t-r+2k}{k} \right)^{\mu} & \text{otherwise}, \end{cases} \tag{3.5}$$

where μ and ν are defined by

$$\begin{aligned} \mu &= \frac{pq-p-q}{p} + \frac{2(p-q)}{p(pq-1)}, \\ \nu &= \frac{q(p-1)}{p(pq-1)}, \end{aligned} \tag{3.6}$$

and k is that used in Lemma 2.2. χ_1 is a characteristic function of a set

$$S_1 = \{(x, t) \in \mathbf{R}^3 \times [0, T] : -k \leqslant t - |x| \leqslant k\} \tag{3.7}$$

and χ_2 is a characteristic function of a set

$$S_2 = \{(x, t) \in \mathbf{R}^3 \times [0, T] : k \leqslant t - |x|\}. \tag{3.8}$$

Remark 3.2. We note that $p \geqslant 2$ implies $\mu \geqslant 0$ because $pq - p - q = q(p-2) + q - p$.

Proposition 3.1 is proved by the following two *a priori* estimates.

LEMMA 3.3. *Let $n = 3$. Suppose that $2 \leqslant p \leqslant q$. Let (u, v) be a solution of* (2.1) in $\mathbb{R}^3 \times [0, T]$. *Then there exists a positive constant C independent of ε, k, and T such that*

$$\begin{aligned} \|\chi_1 L(|v|^p)\|_1 &\leqslant Ck^2 \, \|\chi_1 v\|_2^p \, D(T), \\ \|\chi_1 L(|u|^q)\|_2 &\leqslant Ck^2 \, \|\chi_1 u\|_1^q \, D(T), \end{aligned} \tag{3.9}$$

for any $T > 0$, where D is defined by

$$D(T) = \begin{cases} \log \dfrac{2T + 3k}{k} & \text{if} \quad p = q = 2, \\ 1 & \text{otherwise.} \end{cases} \tag{3.10}$$

LEMMA 3.4. *Let $n = 3$. Suppose that $2 \leqslant p \leqslant q$. Let (u, v) be a solution of* (2.1) in $\mathbb{R}^3 \times [0, T]$. *Then there exists a positive constant C independent of ε, k, and T such that for any $T > 0$,*

$$\begin{aligned} \|\chi_2 L(|v|^p)\|_1 &\leqslant Ck^2 \{ \|\chi_1 v\|_2^p + \|\chi_2 v\|_2^p \, E_1(T) \}, \\ \|\chi_2 L(|u|^q)\|_2 &\leqslant Ck^2 \{ \|\chi_1 u\|_1^q + \|\chi_2 u\|_1^q \, E_2(T) \}, \end{aligned} \tag{3.11}$$

where E_1 and E_2 are defined by

$$\begin{aligned}
&E_1(T) = E_2(T) = 1 && \text{if} \quad F(p, q) < 0, \\
&E_1(T) = \left(\log \frac{T + 3k}{k} \right)^{1 - p\nu}, && \\
&E_2(T) = \left(\log \frac{T + 3k}{k} \right)^{\nu} && \text{if} \quad F(p, q) = 0 \quad \text{with} \quad p \neq q, \\
&E_1(T) = E_2(T) = \log \frac{T + 2k}{k} && \text{if} \quad F(p, q) = 0 \quad \text{with} \quad p = q, \\
&E_1(T) = \left(\frac{T + 2k}{k} \right)^{p(q-1) F(p, q)}, && \\
&E_2(T) = \left(\frac{T + 2k}{k} \right)^{q(p-1) F(p, q)} && \text{if} \quad F(p, q) > 0.
\end{aligned} \tag{3.12}$$

Remark 3.5. We note that

$$1 - pv = \frac{q-1}{pq-1} > 0. \tag{3.13}$$

First, we shall start from function spaces X defined by

$$X = \{(u, v) \in \{C^2(\mathbf{R}^3 \times [0, T])\}^2 : \\ \operatorname{supp}(u, v) \subset \{|x| \leqslant t + k\}, \|(u, v)\|_X < \infty\}, \tag{3.14}$$

where

$$\|(u, v)\|_X = \sum_{|\alpha| \leqslant 2} (\|\nabla_x^\alpha u\|_1 + \|\nabla_x^\alpha v\|_2). \tag{3.15}$$

Note that $\partial u/\partial t$ and $\partial v/\partial t$ are expressed by $\nabla_x u$ and $\nabla_x v$ in view of the representation formula of the solution. So it is sufficient to consider the spatial derivatives only. We also note that X is a Banach space for any fixed $T > 0$ because it follows from the definition of the norm (3.3) that there exists a positive constant C_T depending on T such that

$$\|u\|_1, \|u\|_2 \geqslant C_T |u(x, t)|, \qquad t \in [0, T]. \tag{3.16}$$

Our purpose is to construct a unique solution in X of the equivalent integral equation (2.1) which must be a classical solution of the original p-q systems.

In order to see this, putting

$$M = \max_{|\alpha| \leqslant 2} \{\|\nabla_x^\alpha u^0\|_1, \|\nabla_x^\alpha v^0\|_2\} > 0, \tag{3.17}$$

we also define a closed subspace Y of X by

$$Y = \{(u, v) \in X : \|\chi_1 \nabla_x^\alpha u\|_1, \|\chi_1 \nabla_x^\alpha v\|_2 \leqslant 2M\varepsilon, \\ \|\chi_2 \nabla_x^\alpha u\|_1 \leqslant N\varepsilon^p, \|\chi_2 \nabla_x^\alpha v\|_2 \leqslant N\varepsilon^q \ (|\alpha| \leqslant 2)\}, \tag{3.18}$$

where N is defined by

$$N = 2Ck^2 \max\{p^2(2M)^p, q^2(2M)^q\} \tag{3.19}$$

and C is that used in the *a priori* estimate (3.9), (3.11). We note that Lemma 2.3 and Lemma 2.4 imply $M < \infty$.

The solution will be constructed by a contraction mapping argument in Y if ε is suitably small. After two solutions are constructed in each domain, S_1 and S_2, we know that one must coincide with the other at the intersection of both domains by the uniqueness of the solution.

In the rest of this section, we require Hölder's inequality for the norm $\|\cdot\|_j$ $(j=1, 2)$,

$$\| |u_1|^a \, |u_2|^b \|_j \leqslant \|u_1\|_j^a \, \|u_2\|_j^b, \qquad a+b=1, \quad a, b \in [0, 1]. \tag{3.20}$$

We also denote $\partial/\partial x_i$ by ∂_i for $i=1, 2, 3$, and put

$$\begin{aligned} \partial^l w_m &= \max_{|\alpha| \leqslant l} \{ |\nabla_x^\alpha v_m|, |\nabla_x^\alpha v_{m-1}| \}, \\ \partial^l z_m &= \max_{|\alpha| \leqslant l} \{ |\nabla_x^\alpha u_m|, |\nabla_x^\alpha u_{m-1}| \}, \end{aligned} \qquad l=0, 1, 2, \tag{3.21}$$

where ∂^0 will be omitted and we write $\partial = \partial^1$.

Construction of a Classical Solution in S_1

For simplicity, we shall omit the notation χ_1 and write $L\cdot$ instead of $L(\cdot)$ if possible.

Convergence of $\{(u_m, v_m)\}$ in S_1. Taking the norm of both sides of the iteration frame (3.2), we have, by the *a priori* estimates (3.9),

$$\begin{aligned} \|u_m\|_1 &\leqslant \|u_0\|_1 + \|L(w_{m-1})^p\|_1 \\ &\leqslant M\varepsilon + Ck^2 \, \|w_{m-1}\|_2^p \, D(T) \end{aligned} \tag{3.22}$$

and, similarly,

$$\|v_m\|_2 \leqslant M\varepsilon + Ck^2 \, \|z_{m-1}\|_2^q \, D(T). \tag{3.23}$$

The last inequalities show that $\|u_m\|_1$, $\|v_m\|_2 \leqslant 2M\varepsilon$ $(m \in \mathbf{N})$ provided

$$Ck^2 (2M\varepsilon)^p \, D(T), \; Ck^2 (2M\varepsilon)^q \, D(T) \leqslant M\varepsilon. \tag{3.24}$$

Next, we shall estimate the differences under (3.24). The iteration frame gives us

$$\begin{aligned} \|u_{m+1} - u_m\|_1 &= \|L(|v_m|^p - |v_{m-1}|^p)\|_1 \\ &\leqslant p \|L((w_m)^{p-1} \, |v_m - v_{m-1}|)\|_1. \end{aligned} \tag{3.25}$$

Hölder's inequality (3.20) and the *a priori* estimate (3.9) yield that

$$\begin{aligned} \|u_{m+1} - u_m\|_1 &\leqslant pCk^2 D(T) \, \|(w_m)^{(p-1)/p} \, |v_m - v_{m-1}|^{1/p}\|_2^p \\ &\leqslant pCk^2 D(T) \, \|w_m\|_2^{p-1} \, \|v_m - v_{m-1}\|_2. \end{aligned} \tag{3.26}$$

Similarly, we get

$$\|v_{m+1} - v_m\|_2 \leqslant qCk^2 D(T) \, \|z_m\|_1^{q-1} \, \|u_m - u_{m-1}\|_1. \tag{3.27}$$

Therefore a convergence of $\{(u_m, v_m)\}$ follows from

$$\begin{aligned} \|u_{m+2}-u_{m+1}\|_1 &\leqslant 4^{-1}\|u_m-u_{m-1}\|_1, \\ \|v_{m+2}-v_{m+1}\|_2 &\leqslant 4^{-1}\|v_m-v_{m-1}\|_2 \end{aligned} \tag{3.28}$$

provided

$$pqC^2k^4(2M\varepsilon)^{p+q-2}D^2(T)\leqslant 4^{-1}. \tag{3.29}$$

In fact, we obtain

$$\begin{cases} \|u_m-u_{m-1}\|_1\leqslant \dfrac{N_0}{2^m}, \\ \|v_m-v_{m-1}\|_2\leqslant \dfrac{N_0}{2^m}, \end{cases} \tag{3.30}$$

where $N_0>0$ is independent of m defined by

$$N_0=\max\{2\|u_1-u_0\|_1, 4\|u_2-u_1\|_1, 2\|v_1-v_0\|_2, 4\|v_2-v_1\|_2\}. \tag{3.31}$$

Convergence of $\{(\partial_i u_m, \partial_i v_m)\}$ $(i=1, 2, 3)$ *in* S_1. Assume that (3.24) and (3.29) hold. Applying ∂_i to (3.2), we have

$$\begin{aligned} \partial_i u_m &= \partial_i u_0 + pL(|v_{m-1}|^{p-1}\partial_i v_{m-1}), \\ \partial_i v_m &= \partial_i v_0 + qL(|u_{m-1}|^{q-1}\partial_i u_{m-1}). \end{aligned} \tag{3.32}$$

Taking the norm of both sides, we have, by the *a priori* estimates (3.9),

$$\begin{aligned} \|\partial_i u_m\|_1 &\leqslant \|\partial_1 u_0\|_1 + p\|L(\partial w_{m-1})^p\|_1 \\ &\leqslant M\varepsilon + pCk^2D(T)\|\partial w_{m-1}\|_2^p \end{aligned} \tag{3.33}$$

and, similarly,

$$\|\partial_i v_m\|_2 \leqslant M\varepsilon + qCk^2D(T)\|\partial z_{m-1}\|_1^q. \tag{3.34}$$

The last two inequalities show that $\|\partial_i u_m\|_1$, $\|\partial_i v_m\|_2 \leqslant 2M\varepsilon$ $(m\in\mathbf{N})$ provided

$$pCk^2(2M\varepsilon)^p D(T), qCk^2(2M\varepsilon)^q D(T)\leqslant M\varepsilon. \tag{3.35}$$

This is stronger than (3.24).

Next, we shall estimate the differences under (3.29) and (3.35). The iteration frame (3.32) gives us

$$\partial_i u_{m+1} - \partial_i u_m = pL(|v_m|^{p-1}\,\partial_i v_m - |v_{m-1}|^{p-1}\,\partial_i v_{m-1}). \tag{3.36}$$

The differentiablity of a function $|\cdot|^{p-1}$ yields

$$\begin{aligned} &\big|\,|v_m|^{p-1}\,\partial_i v_m - |v_{m-1}|^{p-1}\,\partial_i v_{m-1}\big| \\ &\quad \leqslant \big|\,|v_m|^{p-1} - |v_{m-1}|^{p-1}\big|\,|\partial_i v_m| + |v_{m-1}|^{p-1}\,|\partial_i v_m - \partial_i v_{m-1}| \\ &\quad \leqslant (p-1)(\partial w_m)^{p-1}\,|v_m - v_{m-1}| + (w_{m-1})^{p-1}\,|\partial_i v_m - \partial_i v_{m-1}|. \end{aligned} \tag{3.37}$$

Hence we have

$$\begin{aligned} \|\partial_i u_{m+1} - \partial_i u_m\|_1 \leqslant{}& p(p-1)\,Ck^2 D(T)\,\|\partial w_m\|_2^{p-1}\,\|v_m - v_{m-1}\|_2 \\ &+ pCk^2 D(T)\,\|w_{m-1}\|_2^{p-1}\,\|\partial_i v_m - \partial_i v_{m-1}\|_2. \end{aligned} \tag{3.38}$$

It follows from (3.30) that

$$\|\partial_i u_{m+1} - \partial_i u_m\|_1 \leqslant \frac{N_{1,p}}{2^m} + pCk^2(2M\varepsilon)^{p-1}\,D(T)\,\|\partial_i v_m - \partial_i v_{m-1}\|_2, \tag{3.39}$$

where $N_{1,p} > 0$ is independent of m defined by

$$N_{1,p} = N_0 p(p-1)\,Ck^2(2M\varepsilon)^{p-1}\,D(T). \tag{3.40}$$

Similarly, we obtain

$$\|\partial_i v_{m+1} - \partial_i v_m\|_2 \leqslant \frac{N_{1,q}}{2^m} + qCk^2(2M\varepsilon)^{q-1}\,D(T)\,\|\partial_i u_m - \partial_i u_{m-1}\|_1. \tag{3.41}$$

Therefore a convergence of $\{(\partial_i u_m, \partial_i v_m)\}$ follows from

$$\begin{aligned} \|\partial_i u_{m+2} - \partial_i u_{m+1}\|_1 &\leqslant \frac{N_{1,1}}{2^m} + \frac{1}{4}\,\|\partial_i u_m - \partial_i u_{m-1}\|_1 \\ \|\partial_i v_{m+2} - \partial_i v_{m+1}\|_2 &\leqslant \frac{N_{1,2}}{2^m} + \frac{1}{4}\,\|\partial_i v_m - \partial_i v_{m-1}\|_2, \end{aligned} \tag{3.42}$$

where both positive constants $N_{1,1}$ and $N_{1,2}$ are independent of m and defined by

$$N_{1,1} = 2^{-1} N_{1,p} + pCk^2 (2M\varepsilon)^{p-1} D(T) N_{1,q},$$
$$N_{1,2} = 2^{-1} N_{1,q} + qCk^2 (2M\varepsilon)^{q-1} D(T) N_{1,p}, \tag{3.43}$$

provided (3.29) holds. In fact, we obtain

$$\|\partial_i u_m - \partial_i u_{m-1}\|_1 \leqslant \frac{mN_1}{2^m}$$
$$\|\partial_i v_m - \partial_i v_{m-1}\|_1 \leqslant \frac{mN_1}{2^m}, \tag{3.44}$$

where $N_1 > 0$ is independent of m defined by

$$N_1 = 2 \max\{N_{1,1} + \|\partial_i u_1 - \partial_i u_0\|_1, N_{1,1} + 2\|\partial_i u_2 - \partial_i u_1\|_1,$$
$$N_{1,2} + \|\partial_i v_1 - \partial_i v_0\|_2, N_{1,2} + 2\|\partial_i v_2 - \partial_i v_1\|_2\}. \tag{3.45}$$

Convergence of $\{(\partial_i \partial_j u_m, \partial_i \partial_j v_m)\}$ $(i, j = 1, 2, 3)$ *in* S_1. Assume that (3.29) and (3.35) hold. Applying ∂_j to (3.32), we have

$$\partial_i \partial_j u_m = \partial_i \partial_j u_0 + pL((p-1)\,|v_{m-1}|^{p-2}\,\partial_i v_{m-1}\,\partial_j v_{m-1}$$
$$+ |v_{m-1}|^{p-1}\,\partial_i \partial_j v_{m-1}),$$
$$\partial_i \partial_j v_m = \partial_i \partial_j v_0 + qL((q-1)\,|u_{m-1}|^{q-2}\,\partial_i u_{m-1}\,\partial_j u_{m-1}$$
$$+ |u_{m-1}|^{q-1}\,\partial_i \partial_j u_{m-1}), \tag{3.46}$$

Taking the norm of both sides, we have, by the *a priori* estimates (3.9),

$$\|\partial_i \partial_j u_m\|_1 \leqslant \|\partial_i \partial_j u_0\|_1 + p^2 \|L(\partial^2 w_{m-1})^p\|_1$$
$$\leqslant M\varepsilon + p^2 Ck^2 D(T) \|\partial^2 w_{m-1}\|_2^p \tag{3.47}$$

and, similarly,

$$\|\partial_i \partial_j v_m\|_2 \leqslant M\varepsilon + q^2 Ck^2 D(T) \|\partial^2 z_{m-1}\|_1^q. \tag{3.48}$$

The last two inequalities show that $\|\partial_i \partial_j u_m\|_1$, $\|\partial_i \partial_j v_m\|_2 \leqslant 2M\varepsilon$ $(m \in \mathbf{N})$ provided

$$p^2 Ck^2 (2M\varepsilon)^p D(T),\ q^2 Ck^2 (2M\varepsilon)^q D(T) \leqslant M\varepsilon. \tag{3.49}$$

This is stronger than (3.35).

Next, we shall estimate the differences under (3.29) and (3.49). The iteration frame (3.46) gives us

$$\begin{aligned}\partial_i \partial_j u_{m+1} - \partial_i \partial_j u_m = p(p-1)\, L\{ (|v_m|^{p-2} - |v_{m-1}|^{p-2})\, \partial_i v_m\, \partial_j v_m \\ + |v_{m-1}|^{p-2}\, (\partial_i v_m - \partial_i v_{m-1})\, \partial_j v_m \\ + |v_{m-1}|^{p-2}\, \partial_i v_{m-1} (\partial_j v_m - \partial_j v_{m-1})\} \\ + pL\{ (|v_m|^{p-1} - |v_{m-1}|^{p-1})\, \partial_i \partial_j v_m \\ + |v_{m-1}|^{p-1}\, (\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1})\}. \qquad (3.50)\end{aligned}$$

Taking the norm of both sides, we have, by the Hölder continuity of a function $|\cdot|^{p-2}$ when $2 \leqslant p \leqslant 3$,

$$\begin{aligned}\|\partial_i \partial_j u_{m+1} - \partial_i \partial_j u_m\|_1 \leqslant p(p-1) \{ \|L((\partial w_m)^2\, |v_m - v_{m-1}|^{p-2}|)\|_1 \\ + \|L((\partial w_m)^{p-1}\, |\partial_i v_m - \partial_i v_{m-1}|)\|_1 \\ + \|L((\partial w_m)^{p-1}\, |\partial_j v_m - \partial_j v_{m-1}|)\|_1 \\ + \|L((\partial^2 w_m)^{p-1}\, |v_m - v_{m-1}|)\|_1\} \\ + p\, \|L((w_m)^{p-1}\, |\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}|)\|_1. \\ (3.51)\end{aligned}$$

Equations (3.9) and (3.20) yield

$$\begin{aligned}&\|\partial_i \partial_j u_{m+1} - \partial_i \partial_j u_m\|_1 \\ &\quad \leqslant p(p-1)\, Ck^2 D(T) \{ \|\partial w_m\|_2^2\, \|v_m - v_{m-1}\|_2^{p-2} \\ &\qquad + \|\partial w_m\|_2^{p-1}\, \|\partial_i v_m - \partial_i v_{m-1}\|_2 + \|\partial w_m\|_2^{p-1}\, \|\partial_j v_m - \partial_j v_{m-1}\|_2 \\ &\qquad + \|\partial^2 w_m\|_2^{p-1}\, \|v_m - v_{m-1}\|_2\} \\ &\qquad + pCk^2 D(T)\, \|w_m\|_2^{p-1}\, \|\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}\|_2. \qquad (3.52)\end{aligned}$$

We also get by the differentiability of a function $|\cdot|^{p-2}$ when $p > 3$,

$$\begin{aligned}&\|\partial_i \partial_j u_{m+1} - \partial_i \partial_j u_m\|_1 \\ &\quad \leqslant p(p-1)\, Ck^2 D(T) \{ (p-2)\, \|\partial w_m\|_2^{p-1}\, \|v_m - v_{m-1}\|_2 \\ &\qquad + \|\partial w_m\|_2^{p-1}\, \|\partial_i v_m - \partial_i v_{m-1}\|_2 + \|\partial w_m\|_2^{p-1}\, \|\partial_j v_m - \partial_j v_{m-1}\|_2 \\ &\qquad + \|\partial^2 w_m\|_2^{p-1}\, \|v_m - v_{m-1}\|_2\} \\ &\qquad + pCk^2 D(T)\, \|w_m\|_2^{p-1}\, \|\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}\|_2. \qquad (3.53)\end{aligned}$$

Combining the two cases together with (3.30) and (3.44), we obtain

$$\|\partial_i \partial_j u_{m+1} - \partial_i \partial_j u_m\|_1 \leqslant \frac{mN_{2,\,p}}{2^{\min\{1,\,p-2\}m}} + pCk^2(2M\varepsilon)^{p-1}\, D(T)\, \|\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}\|_2, \tag{3.54}$$

where $N_{2,\,p} > 0$ is independent of m defined by

$$N_{2,\,p} = p(p-1)\, Ck^2 D(T) \begin{cases} (2M\varepsilon)^2\, N_0^{p-2} + (2M\varepsilon)^{p-1}\,(N_0 + 2N_1) \\ \qquad \text{when} \quad 2 \leqslant p \leqslant 3, \\ (2M\varepsilon)^{p-1}\,((p-1)\, N_0 + 2N_1) \\ \qquad \text{when} \quad p > 3. \end{cases} \tag{3.55}$$

Similarly, we have

$$\|\partial_i \partial_j v_{m+1} - \partial_i \partial_j v_m\|_2 \leqslant \frac{mN_{2,\,q}}{2^{\min\{1,\,q-2\}m}} + qCk^2(2M\varepsilon)^{q-1}\, D(T)\, \|\partial_i \partial_j u_m - \partial_i \partial_j u_{m-1}\|_1. \tag{3.56}$$

Therefore a convergence of $\{(\partial_i \partial_j u_m, \partial_i \partial_j v_m)\}$ follows from

$$\begin{cases} \|\partial_i \partial_j u_{m+2} - \partial_i \partial_j u_{m+1}\|_1 \leqslant \dfrac{mN_{2,\,1}}{2^{\min\{1,\,p-2\}\,m}} + \dfrac{1}{4}\, \|\partial_i \partial_j u_m - \partial_i \partial_j u_{m-1}\|_1 \\ \|\partial_i \partial_j v_{m+2} - \partial_i \partial_j v_{m+1}\|_2 \leqslant \dfrac{mN_{2,\,2}}{2^{\min\{1,\,p-2\}m}} + \dfrac{1}{4}\, \|\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}\|_2, \end{cases} \tag{3.57}$$

where both positive constants $N_{2,\,1}$ and $N_{2,\,2}$ are independent of m defined by

$$\begin{aligned} N_{2,\,1} &= 2^{1-\min\{1,\,p-2\}} N_{2,\,p} + pCk^2(2M\varepsilon)^{p-1}\, D(T)\, N_{2,\,q}, \\ N_{2,\,2} &= 2^{1-\min\{1-p-2\}}\, N_{2,\,q} + qCk^2(2M\varepsilon)^{q-1}\, D(T)\, N_{2,\,p}, \end{aligned} \tag{3.58}$$

provided (3.29) holds. In fact, we obtain

$$\begin{aligned} \|\partial_i \partial_j u_m - \partial_i \partial_j u_{m-1}\|_1 &\leqslant \frac{m^2 N_2}{2^{\min\{1,\,p-2\}\,m}} \\ \|\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}\|_2 &\leqslant \frac{m^2 N_2}{2^{\min\{1,\,p-2\}\,m}}, \end{aligned} \tag{3.59}$$

where $N_2 > 0$ is independent of m defined by

$$N_2 = \max\{N_{2,1} + 2\,\|\partial_i \partial_j u_1 - \partial_i \partial_j u_0\|_1,\ N_{2,1} + 4\,\|\partial_i \partial_j u_2 - \partial_i \partial_j u_1\|_1,$$
$$N_{2,2} + 2\,\|\partial_i \partial_j v_1 - \partial_i \partial_j v_0\|_2,\ N_{2,2} + 4\,\|\partial_i \partial_j v_2 - \partial_i \partial_j v_1\|_2\}. \tag{3.60}$$

Hence we can conclude that (3.29) and (3.49) make a classical solution in S_1 of the system of integral equations (2.1) which is also a classical solution in S_1 of the original problem (1.1), (1.2).

Construction of a Classical Solution in S_2

The proof will be done in almost the same way as that in the case of S_1. Here and hereafter we assume the conditions (3.29) and (3.49) to ensure the existence of a classical solution in S_1. Recall Lemma 2.3.

Convergence of $\{(u_m, v_m)\}$ *in* S_2. Taking the norm of both sides of the iteration frame (3.2), we have, by the *a priori* estimates (3.11) and Lemma 2.3,

$$\begin{aligned}\|\chi_2 u_m\|_1 &\leqslant \|\chi_2 L(w_{m-1})^p\|_1\\ &\leqslant Ck^2\{\|\chi_1 w_{m-1}\|_2^p + \|\chi_2 w_{m-1}\|_2^p\, E_1(T)\}\end{aligned} \tag{3.61}$$

and, similarly,

$$\|\chi_2 v_m\|_2 \leqslant Ck^2\{\|\chi_1 z_{m-1}\|_1^p + \|\chi_2 z_{m-1}\|_1^p\, E_2(T)\}. \tag{3.62}$$

The last inequalities show that $\|\chi_2 u_m\|_1 \leqslant N\varepsilon^p$, $\|\chi_2 v_m\|_2 \leqslant N\varepsilon^q$ $(m \in \mathbf{N})$ provided

$$\begin{aligned}Ck^2\{(2M\varepsilon)^p + (N\varepsilon^q)^p\, E_1(T)\} &\leqslant N\varepsilon^p,\\ Ck^2\{(2M\varepsilon)^q + (N\varepsilon^p)^q\, E_2(T)\} &\leqslant N\varepsilon^q.\end{aligned} \tag{3.63}$$

Next, we shall estimate the differences under (3.63). The iteration frame gives us

$$\|\chi_2 u_{m+1} - \chi_2 u_m\|_1 \leqslant p\,\|\chi_2 L((w_m)^{p-1}\,|v_m - v_{m-1}|)\|_1. \tag{3.64}$$

Hölder's inequality (3.20) and the *a priori* estimate (3.11) yield that

$$\begin{aligned}\|\chi_2 u_{m+1} - \chi_2 u_m\|_1 \leqslant pCk^2\{&\|\chi_1 w_m\|_2^{p-1}\,\|\chi_1 v_m - \chi_1 v_{m-1}\|_2\\ &+ \|\chi_2 w_m\|_2^{p-1}\,\|\chi_2 v_m - \chi_2 v_{m-1}\|_2\, E_1(T)\}.\end{aligned} \tag{3.65}$$

Hence (3.29) and (3.49) mean that (3.30) makes

$$\|\chi_2 u_{m+1} - \chi_2 u_m\|_1 \leqslant \frac{pCk^2(2M\varepsilon)^{p-1} N_0}{2^m} + pCk^2 \|\chi_2 w_m\|_2^{p-1} \|\chi_2 v_m - \chi_2 v_{m-1}\|_2 E_1(T). \tag{3.66}$$

Similarly, we get

$$\|\chi_2 v_{m+1} - \chi_2 v_m\|_2 \leqslant \frac{qCk^2(2M\varepsilon)^{q-1} N_0}{2^m} + qCk^2 \|\chi_2 z_m\|_1^{q-1} \|\chi_2 u_m - \chi_2 u_{m-1}\|_1 E_2(T). \tag{3.67}$$

Therefore a convergence of $\{(u_m, v_m)\}$ follows from

$$\begin{aligned} \|\chi_2 u_{m+2} - \chi_2 u_{m+1}\|_1 &\leqslant \frac{N_{3,1}}{2^m} + \frac{1}{4} \|\chi_2 u_m - \chi_2 u_{m-1}\|_1, \\ \|\chi_2 v_{m+2} - \chi_2 v_{m+1}\|_2 &\leqslant \frac{N_{3,2}}{2^m} + \frac{1}{4} \|\chi_2 v_m - \chi_2 v_{m-1}\|_2, \end{aligned} \tag{3.68}$$

where

$$\begin{aligned} N_{3,1} &= 2^{-1} pCk^2(2M\varepsilon)^{p-1} N_0 + pqC^2k^4(2M\varepsilon)^{q-1} (N\varepsilon^q)^{p-1} N_0 E_1(T), \\ N_{3,2} &= 2^{-1} qCk^2(2M\varepsilon)^{q-1} N_0 + pqC^2k^4(2M\varepsilon)^{p-1} (N\varepsilon^p)^{q-1} N_0 E_2(T), \end{aligned} \tag{3.69}$$

provided

$$pqC^2k^4(N\varepsilon^q)^{p-1} (N\varepsilon^p)^{q-1} E_1(T) E_2(T) \leqslant 4^{-1}. \tag{3.70}$$

In fact, we obtain

$$\begin{aligned} \|\chi_2 u_m - \chi_2 u_{m-1}\|_1 &\leqslant \frac{mN_3}{2^m}, \\ \|\chi_2 v_m - \chi_2 v_{m-1}\|_2 &\leqslant \frac{mN_3}{2^m}, \end{aligned} \tag{3.71}$$

where $N_3 > 0$ is independent of m defined by

$$N_3 = 2\max\{N_{3,1} + \|\chi_2 u_1 - \chi_2 u_0\|_1, N_{3,1} + 2\|\chi_2 u_2 - \chi_2 u_1\|_1,$$
$$N_{3,2} + \|\chi_2 v_1 - \chi_2 v_0\|_2, N_{3,2} + 2\|\chi_2 v_2 - \chi_2 v_1\|_2\}. \tag{3.72}$$

Convergence of $\{(\partial_i u_m, \partial_i v_m)\}$ $(i = 1, 2, 3)$ *in* S_2. Assume that (3.63) and (3.70) hold. Taking the norm of both sides in (3.32), we have, by the *a priori* estimates (3.11),

$$\|\chi_2 \partial_i u_m\|_1 \leqslant p\|\chi_2 L(\partial w_{m-1})^p\|_1$$
$$\leqslant pCk^2\{\|\chi_1 \partial w_{m-1}\|_2^p + \|\chi_2 \partial w_{m-1}\|_2^p E_1(T)\} \tag{3.73}$$

and, similarly,

$$\|\chi_2 \partial_i v_m\|_2 \leqslant pCk^2\{\|\chi_1 \partial z_{m-1}\|_1^p + \|\chi_2 \partial z_{m-1}\|_1^p E_2(T)\}. \tag{3.74}$$

The last two inequalities show that $\|\chi_2 \partial_i u_m\|_1 \leqslant N\varepsilon^p$, $\|\chi_2 \partial_i v_m\|_2 \leqslant N\varepsilon^q$ $(m \in \mathbf{N})$, provided

$$\begin{aligned} pCk^2\{(2M\varepsilon)^p + (N\varepsilon^q)^p E_1(T)\} &\leqslant N\varepsilon^p, \\ qCk^2\{(2M\varepsilon)^q + (N\varepsilon^p)^q E_2(T)\} &\leqslant N\varepsilon^q. \end{aligned} \tag{3.75}$$

This is stronger than (3.63).

Next, we shall estimate the differences under (3.75) and (3.70). Similarly to the case of S_1, we have

$$\begin{aligned} \|\chi_2 \partial_i u_{m+1} - \chi_2 \partial_i u_m\|_1 \leqslant{}& p(p-1)\, Ck^2\{\|\chi_1 \partial w_m\|_2^{p-1} \|\chi_1 v_m - \chi_1 v_{m-1}\|_2 \\ &+ \|\chi_2 \partial w_m\|_2^{p-1} \|\chi_2 v_m - \chi_2 v_{m-1}\|_2 E_1(T)\} \\ &+ pCk^2\{\|\chi_1 w_{m-1}\|_2^{p-1} \|\chi_1 \partial_i v_m - \chi_1 \partial_i v_{m-1}\|_2 \\ &+ \|\chi_2 w_{m-1}\|_2^{p-1} \|\chi_2 \partial_i v_m - \chi_2 \partial_i v_{m-1}\|_2 E_1(T)\}. \end{aligned} \tag{3.76}$$

It follows from (3.30), (3.44), and (3.71) that

$$\|\chi_2 \partial_i u_{m+1} - \chi_2 \partial_i u_m\|_1$$
$$\leqslant \frac{mN_{4,p,q}}{2^m} + pCk^2 (N\varepsilon^q)^{p-1} E_1(T)\, \|\chi_2 \partial_i v_m - \chi_2 \partial_i v_{m-1}\|_2, \tag{3.77}$$

where $N_{4,\,p,\,q}>0$ is independent of m defined by

$$N_{4,\,p,\,q}=\frac{N_{1,\,p}}{D(T)}+pCk^2(2M\varepsilon)^{p-1}N_1+p(p-1)\,Ck^2(N\varepsilon^q)^{p-1}N_3E_1(T)\}. \tag{3.78}$$

Similarly, we obtain

$$\begin{aligned}\|\chi_2\,\partial_i v_{m+1}-\chi_2\,\partial_i v_m\|_2&\\ &\leqslant\frac{mN_{4,\,q,\,p}}{2^m}+qCk^2(N\varepsilon^p)^{q-1}E_2(T)\,\|\chi_2\,\partial_i u_m-\chi_2\,\partial_i u_{m-1}\|_1.\end{aligned} \tag{3.79}$$

Note that E_1 is replaced by E_2 in the definition of $N_{4,\,q,\,p}$. Therefore a convergence of $\{(\chi_2\,\partial_i u_m,\chi_2\,\partial_i v_m)\}$ follows from

$$\begin{aligned}\|\chi_2\,\partial_i u_{m+2}-\chi_2\,\partial_i u_{m+1}\|_1&\leqslant\frac{mN_{4,\,1}}{2^m}+\frac{1}{4}\|\chi_2\,\partial_i u_m-\chi_2\,\partial_i u_{m-1}\|_1,\\ \|\chi_2\,\partial_i v_{m+2}-\chi_2\,\partial_i v_{m+1}\|_2&\leqslant\frac{mN_{4,\,2}}{2^m}+\frac{1}{4}\|\chi_2\,\partial_i v_m-\chi_2\,\partial_i v_{m-1}\|_2,\end{aligned} \tag{3.80}$$

where both positive constants $N_{4,\,1}$ and $N_{4,\,2}$ are independent of m defined by

$$\begin{aligned}N_{4,\,1}&=2^{-1}N_{4,\,p,\,q}+pCk^2(N\varepsilon^q)^{p-1}E_1(T)\,N_{4,\,q,\,p},\\ N_{4,\,2}&=2^{-1}N_{4,\,q,\,p}+qCk^2(N\varepsilon^p)^{q-1}E_2(T)\,N_{4,\,p,\,q},\end{aligned} \tag{3.81}$$

provided (3.70) holds. In fact, we obtain

$$\begin{aligned}\|\chi_2\,\partial_i u_m-\chi_2\,\partial_i u_{m-1}\|_1&\leqslant\frac{m^2N_4}{2^m}\\ \|\chi_2\,\partial_i v_m-\chi_2\,\partial_i v_{m-1}\|_2&\leqslant\frac{m^2N_4}{2^m},\end{aligned} \tag{3.82}$$

where $N_4>0$ is independent of m defined by

$$\begin{aligned}N_4=\max\{&N_{4,\,1}+2\,\|\chi_2\,\partial_i u_1-\chi_2\,\partial_i u_0\|_1,\ N_{4,\,1}+4\,\|\chi_2\,\partial_i u_2-\chi_2\,\partial_i u_1\|_1,\\ &N_{4,\,2}+2\,\|\chi_2\,\partial_i v_1-\chi_2\,\partial_i v_0\|_2,\ N_{4,\,2}+4\,\|\chi_2\,\partial_i v_2-\chi_2\,\partial_i v_1\|_2\}.\end{aligned} \tag{3.83}$$

Convergence of $\{(\chi_2 \partial_i \partial_j u_m, \chi_2 \partial_i \partial_j v_m)\}$ $(i, j = 1, 2, 3)$ *in* S_1. Assume that (3.70) and (3.75) hold. In the same way as S_1, we have, by the *a priori* estimates (3.11),

$$\begin{aligned}\|\chi_2 \partial_i \partial_j u_m\|_1 &\leqslant p^2 \|\chi_2 L(\partial_2 w_{m-1})^p\|_1 \\ &\leqslant p^2 C k^2 \{\|\chi_1 \partial^2 w_{m-1}\|_2^p + \|\chi_2 \partial^2 w_{m-1}\|_2^p E_1(T)\}. \end{aligned} \tag{3.84}$$

and, similarly,

$$\|\chi_2 \partial_i \partial_j v_m\|_2 \leqslant q^2 C k^2 \{\|\chi_1 \partial^2 z_{m-1}\|_1^q + \|\chi_2 \partial^2 z_{m-1}\|_1^q E_2(T)\}. \tag{3.85}$$

The last two inequalities show that $\|\chi_2 \partial_i \partial_j u_m\|_1 \leqslant N\varepsilon^p$, $\|\chi_2 \partial_i \partial_j v_m\|_2 \leqslant N\varepsilon^q$ $(m \in \mathbf{N})$, provided

$$\begin{aligned} p^2 C k^2 \{(2M\varepsilon)^p + (N\varepsilon^q)^p E_1(T)\} &\leqslant N\varepsilon^p, \\ q^2 C k^2 \{(2M\varepsilon)^q + (N\varepsilon^p)^q E_2(T)\} &\leqslant N\varepsilon^q. \end{aligned} \tag{3.86}$$

This is stronger than (3.75).

Next, we shall estimate the differences under (3.70) and (3.86). According to the case of S_1, we have, by (3.30), (3.44), (3.59), (3.71), and (3.82), that

$$\begin{aligned}&\|\chi_2 \partial_i \partial_j u_{m+1} - \chi_2 \partial_i \partial_j u_m\|_1 \\ &\quad \leqslant \frac{m^2 N_{5,p,q}}{2^{\min\{1, p-2\} m}} + p C k^2 (N\varepsilon^q)^{p-1} E_1(T) \|\partial_i \partial_j v_m - \partial_i \partial_j v_{m-1}\|_2, \end{aligned} \tag{3.87}$$

where $N_{5,p,q} > 0$ is independent of m defined by

$$\begin{aligned} N_{5,p,q} &= N_{2,p}/D(T) + p C k^2 (2M\varepsilon)^{p-1} N_2 + p(p-1)\, C k^2 E_1(T) \\ &\quad \times \begin{cases} (N\varepsilon^q)^2 N_3^{p-2} + (N\varepsilon^q)^{p-1} (N_3 + 2N_4) & \text{when} \quad 2 \leqslant p \leqslant 3, \\ (N\varepsilon^q)^{p-1} ((p-1) N_3 + 2N_4) & \text{when} \quad p > 3. \end{cases} \end{aligned} \tag{3.88}$$

Similarly, we have

$$\begin{aligned}&\|\chi_2 \partial_i \partial_j v_{m+1} - \chi_2 \partial_i \partial_j v_m\|_2 \\ &\quad \leqslant \frac{m^2 N_{5,q,p}}{2^{\min\{1, q-2\} m}} + q C k^2 (N\varepsilon^p)^{q-1} E_2(T) \|\partial_i \partial_j u_m - \partial_i \partial_j u_{m-1}\|_1. \end{aligned} \tag{3.89}$$

Note that E_1 is replaced by E_2 in the definition of $N_{5,q,p}$. Therefore a convergence of $\{(\chi_2 \partial_i \partial_j u_m, \chi_2 \partial_i \partial_j v_m)\}$ follows from

$$\|\chi_2 \partial_i \partial_j u_{m+2} - \chi_2 \partial_i \partial_j u_{m+1}\|_1 \leqslant \frac{m^2 N_{5,1}}{2^{\min\{1,p-2\}m}} + \frac{1}{4} \|\chi_2 \partial_i \partial_j u_m - \chi_2 \partial_i \partial_j u_{m-1}\|_1$$

$$\|\chi_2 \partial_i \partial_j v_{m+2} - \chi_2 \partial_i \partial_j v_{m+1}\|_1 \leqslant \frac{m^2 N_{5,2}}{2^{\min\{1,p-2\}m}} + \frac{1}{4} \|\chi_2 \partial_i \partial_j v_m - \chi_2 \partial_i \partial_j v_{m-1}\|_2, \tag{3.90}$$

where both positive constants $N_{5,1}$ and $N_{5,2}$ are independent of m and defined by

$$\begin{aligned} N_{5,1} &= 2^{2-\min\{1,p-2\}} N_{5,p,q} + pCk^2 (N\varepsilon^q)^{p-1} E_1(T) N_{5,q,p}, \\ N_{5,2} &= 2^{2-\min\{1,p-2\}} N_{5,q,p} + qCk^2 (N\varepsilon^p)^{q-1} E_2(T) N_{5,p,q}, \end{aligned} \tag{3.91}$$

provided (3.70) holds. In fact, we obtain

$$\begin{aligned} \|\chi_2 \partial_i \partial_j u_m - \chi_2 \partial_i \partial_j u_{m-1}\|_1 &\leqslant \frac{m^3 N_5}{2^{\min\{1,p-2\}m}} \\ \|\chi_2 \partial_i \partial_j v_m - \chi_2 \partial_i \partial_j v_{m-1}\|_2 &\leqslant \frac{m^3 N_5}{2^{\min\{1,p-2\}m}}, \end{aligned} \tag{3.92}$$

where $N_5 > 0$ is independent of m and defined by

$$\begin{aligned} N_5 = \max\{ & N_{5,1} + 2\, \|\chi_2 \partial_i \partial_j u_1 - \chi_2 \partial_i \partial_j u_0\|_1, N_{5,1} \\ & + 4\, \|\chi_2 \partial_i \partial_j u_2 - \chi_2 \partial_i \partial_j u_1\|_1, \\ & N_{5,2} + 2\, \|\chi_2 \partial_i \partial_j v_1 - \chi_2 \partial_i \partial_j v_0\|_2, N_{5,2} \\ & + 4\, \|\chi_2 \partial_i \partial_j v_2 - \chi_2 \partial_i \partial_j v_1\|_2\}. \end{aligned} \tag{3.93}$$

Hence we can conclude that Eqs. (3.70) and (3.86), also Eqs. (3.29) and (3.49), make a classical solution in S_2 of the system of integral equations (2.1) which is also a classical solution in S_2 of the original problem (1.1), (1.2).

Proof of Proposition 3.1. Recall the definition of $D(T)$, (3.10). The existence of a classical solution of (2.1) in S_1 follows from (3.29) and (3.49) which are guaranteed by

$$\frac{2T+3k}{k} \leqslant \exp\left(\frac{1}{8k^2 CM} \varepsilon^{-1}\right) \qquad \text{for} \quad p = q = 2 \tag{3.94}$$

and

$$\varepsilon \leqslant \min\{(4pqk^4C^2)^{-1/(p+q-2)}(2M)^{-1},\\ (2^p p^2 k^2 C)^{-1/(p-1)}\, M^{-1}, (2^q q^2 k^2 C)^{-1/(q-1)}\, M^{-1}\} \qquad \text{otherwise.} \tag{3.95}$$

Note that the latter case implies the global existence for small ε.

Next we shall investigate the existence of a classical solution of (2.1) in S_2 which follows from (3.70) and (3.86). Recall the definitions of N, $E_1(T)$, $E_2(T)$, (3.12), and (3.19). Equations (3.70) and (3.86) are guaranteed by the following four conditions.

Case $F(p, q) < 0$.

$$\varepsilon \leqslant K, \tag{3.96}$$

where K is a positive constant defined by

$$K = \min\{(4pqk^4C^2N^{p+q-2})^{-1/\{p(q-1)+q(p-1)\}},\\ (2p^2k^2CN^{p-1})^{-1/p(q-1)}, (2q^2k^2CN^{q-1})^{-1/q(p-1)}\}. \tag{3.97}$$

This implies a global existence for small ε.

Case $F(p, q) = 0$ *with* $p \neq q$.

$$\frac{T+3k}{k} \leqslant \exp(K^{p(pq-1)}\varepsilon^{-p(pq-1)}), \tag{3.98}$$

where K is that used in (3.97).

Case $F(p, q) = 0$ *with* $p = q$.

$$\frac{T+2k}{k} \leqslant \exp\left(\frac{1}{2p^2k^2CN^{p-1}}\,\varepsilon^{-p(p-1)}\right). \tag{3.99}$$

Case $F(p, q) > 0$.

$$\frac{T+2k}{k} \leqslant K^{F(p,q)^{-1}}\varepsilon^{-F(p,q)^{-1}}, \tag{3.100}$$

where K is that used in (3.97).

Now, we have checked the existence time of a C^2 solution of (2.1) in S_2 for all cases, which is less than or equal to the one in S_1. Therefore the uniqueness of a solution on the line $S_3 = S_1 \cap S_2 = \{(x, t) \in \mathbf{R}^3 \times [0, T] : t - |x| = k\}$ completes the proof of Proposition 3.1, In fact, let (u, v), $(\tilde{u}, \tilde{v})$ be classical solutions of (2.1). Then, by (3.2), we have

$$\begin{aligned} |\chi_{S_3} u - \chi_{S_3} \tilde{u}| &\leqslant p \chi_{S_3} L(w^{p-1} |\chi_1 v - \chi_1 \tilde{v}|), \\ |\chi_{S_3} v - \chi_{S_3} \tilde{v}| &\leqslant q \chi_{S_3} L(z^{q-1} |\chi_1 u - \chi_1 \tilde{u}|), \end{aligned} \tag{3.101}$$

where $w = \max\{|\chi_1 v|, |\chi_1 \tilde{v}|\}$ and $z = \max\{|\chi_1 u|, |\chi_1 \tilde{u}|\}$. We note that both (u, v) and $(\tilde{u}, \tilde{v})$ are classical solutions of the original system (1.1) in the interior of S_1. Hence the uniqueness of the solution of (1.1) yields $u \equiv \tilde{u}$, $v \equiv \tilde{v}$ in the interior of S_1, which implies that $\chi_{S_3}(u - \tilde{u}) = 0$ and $\chi_{S_3}(v - \tilde{v}) = 0$. $\chi_{S_3}(\partial_i u - \partial_i \tilde{u}) = 0$, $\chi_{S_3}(\partial_i v - \partial_i \tilde{v}) = 0$, $\chi_{S_3}(\partial_i \partial_j u - \partial_i \partial_j \tilde{u}) = 0$, and $\chi_{S_3}(\partial_i \partial_j v - \partial_i \partial_j \tilde{v}) = 0$ also follow in a similar way. We finally obtain a unique classical solution of (2.1) in $\mathbf{R}^3 \times [0, T]$ which is also a classical solution of (1.1) in $\mathbf{R}^3 \times [0, T]$.

4. A PRIORI ESTIMATE NEAR THE LIGHT CONE

In this section we shall prove Lemma 3.3 which follows, by the definition of the norm (3.3), from the basic estimates

$$\begin{aligned} \chi_1 w_1 L((\chi_1 w_2)^{-p}) &\leqslant C k^2 D(T), \\ \chi_1 w_2 L((\chi_1 w_1)^{-q}) &\leqslant C k^2 D(T). \end{aligned} \tag{4.1}$$

Introducing the characteristic variables

$$\alpha = \tau + \lambda, \qquad \beta = \tau - \lambda, \tag{4.2}$$

and extending a domain of the integral if necessary, we have, by the representation formula (2.1),

$$L((\chi_1 w_i)^{-p})(x, t) \leqslant 8^{-1} I_i(r, t; p) \qquad \text{in } S_1, \tag{4.3}$$

where $i = 1, 2$, $r \equiv |x|$, and

$$I_i(r, t; p) = \frac{1}{r} \int_{-k}^{k} d\beta \int_{|t-r|}^{t+r} (\alpha - \beta)(\chi_1 w_i)^{-p} \left(\frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2} \right) d\alpha. \tag{4.4}$$

Proof of Lemma 3.3. We shall omit the notation χ_1 for simplicity.

Case 1 (The first line of (4.1) and the second line of (4.1) When $q \geqslant p > 2$ or $p = q = 2$). In this case, it is sufficient to regard w_1 and w_2 as

$$w(r, t) = \frac{t + r + 2k}{k} \tag{4.5}$$

because of the fact that

$$1 \leqslant \frac{t - r + 2k}{k} \leqslant 3 \qquad \text{in } S_1. \tag{4.6}$$

First, we shall consider the case $r \geqslant k$ which means

$$r \geqslant \frac{kw(r, t)}{5}. \tag{4.7}$$

Hence (4.1) follows from

$$I_i(r, t; p) \leqslant \frac{5}{w(r, t)} \int_{-k}^{k} d\beta \int_{0}^{t+r} \left(\frac{\alpha + 2k}{k} \right)^{1-p} d\alpha \tag{4.8}$$

and the definition of $D(T)$, (3.10).

Next, we shall consider the case $r \leqslant k$ which means

$$1 \leqslant w(r, t) \leqslant \frac{2r + 3k}{k} \leqslant 5. \tag{4.9}$$

Then we have

$$\begin{aligned} I_i(r, t; p) &\leqslant \frac{1}{r} \int_{-k}^{k} d\beta \int_{|t-r|}^{t+r} (\alpha + k)\, d\alpha \\ &\leqslant \frac{8k^2}{r} \int_{t-r}^{t+r} d\alpha \\ &\leqslant 80k^2 w^{-1}(r, t). \end{aligned} \tag{4.10}$$

Therefore we obtain the desired estimate (4.1).

Case 2 (The second line of (4.1) when $p = 2 < q$). By definition of w_1, we have to estimate

$$I_1(r, t; q) \leqslant \frac{k}{r} \int_{-k}^{k} d\beta \int_{|t-r|}^{t+r} \left(\frac{\alpha + 2k}{k} \right)^{1-q} \left(\log 4 \frac{\alpha + 2k}{|\beta| + 2k} \right)^{q} d\alpha. \tag{4.11}$$

Now, we use the following simple inequality.

LEMMA 4.1. *Let $\delta>0$ be any given constant. Then, it follows that*

$$\log X \leqslant \frac{X^{\delta}}{\delta} \qquad \text{for} \quad X \geqslant 1. \tag{4.12}$$

Proof. One can readily check this lemma by differentiation.

First, we shall consider the case $r \geqslant k$. Taking δ so small $2-q+q\delta<0$, we have, with the help of Lemma 4.1, that

$$\begin{aligned} I_1(r, t; q) &\leqslant \frac{5 \cdot 2^{-q\delta} 4^{q\delta} k}{\delta^q (t+r+2k)} \int_{-k}^{k} d\beta \int_0^{t+r} \left(\frac{\alpha+2k}{k}\right)^{1-q+q\delta} d\alpha \\ &\leqslant \frac{10 \cdot 2^{2-q} 4^{q\delta} k^2}{\delta^q (q-q\delta-2)} \cdot \frac{k}{t+r+2k} \end{aligned} \tag{4.13}$$

which implies (4.1).

Next, we shall consider the case $r \leqslant k$. In this case we have

$$\frac{1}{3} \leqslant \frac{t+r+2k}{|t-r|+2k} \leqslant \frac{5}{2} \tag{4.14}$$

which implies that w_1 an w_2 are equivalent to numerical constants. So, nothing new comes from estimating $I_1(r, t; q)$. The above two cases complete the proof of Lemma 3.3.

5. A PRIORI ESTIMATE IN THE INSIDE OF THE LIGHT CONE

In this proof, each constant C is independent of ε and may change from line to line. As in the previous section, Lemma 3.4 follows from the basic estimates

$$\begin{aligned} &\chi_2 w_1 L((\chi_1 w_2)^{-p}) \leqslant Ck^2, \qquad \chi_2 w_1 L((\chi_2 w_2)^{-p}) \leqslant Ck^2 E_1(T), \\ &\chi_2 w_2 L((\chi_1 w_1)^{-q}) \leqslant Ck^2, \qquad \chi_2 w_1 L((\chi_2 w_1)^{-q}) \leqslant Ck^2 E_2(T). \end{aligned} \tag{5.1}$$

Similar to the previous section, we shall estimate

$$L((w_i)^{-p})(x, t) \leqslant 8^{-1}\{I_i(r, t; p) + J_i(r, t; p)\} \qquad \text{in } S_2, \tag{5.2}$$

where I_i is that used in (4.4) and

$$J_i(r, t; p) = \frac{1}{r} \int_k^{t-r} d\beta \int_{t-r}^{t+r} (\alpha-\beta)(\chi_2 w_i)^{-p} \left(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2}\right) d\alpha. \tag{5.3}$$

Proof of Lemma 3.4. The entire β-integral in I_i is dominated by Ck. So, the first inequality in each line of (5.1) follows in exactly the same way as that used to bundle the α-integral in the second inequality. Recalling the definitions of $F(p, q)$ in (1.3) and μ in (3.6), we will use the key relation

$$\begin{aligned} 1 - p\mu &= p(q-1)\, F(p, q), \\ \mu + 3 + q - pq &= q(p-1)\, F(p, q). \end{aligned} \tag{5.4}$$

Case 1 (The first line of (5.1) when $F(p, q) \neq 0$, $q > 2$, or $F(p, q) = 0$ with $p = q$). By the definition of w_2 we have

$$J_2(r, t; p) \leqslant \frac{k}{r} \int_k^{t-r} \left(\frac{\beta + 2k}{k}\right)^{-p\mu} d\beta \int_{t-r}^{t+r} \left(\frac{\alpha + 2k}{k}\right)^{1-p} d\alpha. \tag{5.5}$$

Then the α-integral is dominated by, in the case $t - r \geqslant r$ which implies $3(t-r) \geqslant t + r$,

$$2r \left(\frac{t-r+2k}{k}\right)^{1-p}$$

$$\leqslant \begin{cases} 6r \left(\dfrac{t+r+2k}{k}\right)^{-1} \left(\dfrac{t-r+2k}{k}\right)^{2-p} & \text{when} \quad p > 2, \\ \dfrac{6r}{\log(4/3)} \left(\dfrac{t+r+2k}{k}\right)^{-1} \log 4\, \dfrac{t+r+2k}{t-r+2k} & \text{when} \quad p = 2, \end{cases} \tag{5.6}$$

or in the case $t - r \leqslant r$,

$$\begin{gathered} \frac{k}{p-2} \left(\frac{t-r+2k}{k}\right)^{2-p} \leqslant \frac{5r}{p-2} \left(\frac{t+r+2k}{k}\right)^{-1} \left(\frac{t-r+2k}{k}\right)^{2-p} \\ \text{when} \quad p > 2, \\ k \log \frac{t+r+2k}{t-r+2k} \leqslant 5r \left(\frac{t+r+2k}{k}\right)^{-1} \log 4\, \frac{t+r+2k}{t-r+2k} \\ \text{when} \quad p = 2, \end{gathered} \tag{5.7}$$

because $r \geqslant (t + r + 2k)/5$ when $r \geqslant k$. When $r \leqslant k$, w_1 is a numerical constant. Hence we obtain

$$\int_{t-r}^{t+r} \left(\frac{\alpha + 2k}{k}\right)^{1-p} d\alpha \leqslant \frac{Cr}{\chi_2 w_1(r, t)} \qquad \text{in } S_2 \tag{5.8}$$

which proves the first inequality of the first line in (5.1). The second inequality follows from the relation (5.4) and the definition of E_1 in (3.12) which shows

$$J_2(r, t; p) \leqslant \frac{Ck^2 E_1(t-r)}{\chi_2 w_1(r, t)} \qquad \text{in } S_2. \tag{5.9}$$

Case 2 (The first line of (5.1) when $F(p, q)=0$ with $p \neq q$). In this case, we have

$$J_2(r, t; p) \leqslant \frac{k}{r} \int_k^{t-r} \left(\frac{\beta+2k}{k}\right)^{-1} \left(\log \frac{\beta+3k}{k}\right)^{-p\nu} d\beta \int_{t-r}^{t+r} \left(\frac{\alpha+2k}{k}\right)^{1-p} d\alpha. \tag{5.10}$$

The α-integral is already estimated in Case 1. Hence, by the definition of E_1, we obtain (5.9) again.

Case 3 (The second line of (5.1) when $q \geqslant p > 2$). By the definition of w_1, we have

$$J_1(r, t; q) \leqslant \frac{k}{r} \int_k^{t-r} \left(\frac{\beta+2k}{k}\right)^{-q(p-2)} d\beta \int_{t-r}^{t+r} \left(\frac{\alpha+2k}{k}\right)^{1-q} d\alpha. \tag{5.11}$$

When $F(p, q) \neq 0$ or $F(p, q)=0$ with $p=q$, a power of α should be broken as

$$1-q = -(1+\mu)+\mu+2-q. \tag{5.12}$$

Here, the definition of μ in (3.6) makes a key relation,

$$\mu+2-q = \frac{(p-q)(pq+1)}{p(pq-1)} \leqslant 0, \tag{5.13}$$

and we get

$$J_1(r, t; q) \leqslant \frac{Ck}{r} \int_k^{t-r} \left(\frac{\beta+2k}{k}\right)^{\mu+2+q-pq} d\beta \int_{t-r}^{t+r} \left(\frac{\alpha+2k}{k}\right)^{-1-\mu} d\alpha. \tag{5.14}$$

The α-integral is already estimated in Case 1 in which $p-2$ is replaced by $\mu > 0$. Hence we obtain the first inequality of the second line in (5.1). The second inequality follows from the key relation (5.4) and

$$J_1(r, t; q) \leqslant \frac{Ck}{\chi_2 w_2(r, t)} \int_k^{t-r} \left(\frac{\beta+2k}{k}\right)^{-1+q(p-1)\,F(p, q)} d\beta \tag{5.15}$$

which show the desired estimate

$$J_1(r, t; q) \leqslant \frac{Ck^2 E_2(t-r)}{\chi_2 w_2(r, t)} \qquad \text{in } S_2. \tag{5.16}$$

When $F(p, q) = 0$ with $p \neq q$, the α-integral is estimated in Case 1 in which p is replaced by q. Hence it follows from

$$1 - q(p-2) = q - 2 - \frac{1}{p} > 0 \tag{5.17}$$

by (5.4) and (5.13) that

$$J_1(r, t; q) \leqslant Ck^2 \left(\frac{t+r+2k}{k}\right)^{-1} \left(\frac{t-r+2k}{k}\right)^{2-q+1-q(p-2)}. \tag{5.18}$$

Equation (5.4) implies that $3 + q - pq = -1/p$ in this case. Therefore we obtain (5.16), putting $(\log((t-r+3k)/k))^{\nu-\nu}$. The first inequality of the second line in (5.1) follows from

$$I_1(r, t; q) \leqslant \frac{Ck^2}{\chi_2 w_2(r, t)} \left(\frac{t-r+2k}{k}\right)^{1/p+2-q} \left(\log \frac{t-r+3k}{k}\right)^{\nu}, \tag{5.19}$$

and Lemma 4.1 with the fact that $1/p + 2 - q < 0$.

Case 4 (The second line of (5.1) when $q > p = 2$). In this case, we note that $F(p, q) = 0$ is valid only for $p \neq q$. By Lemma 4.1 with small δ satisfying $1 - q\delta > 0$ and $2 - q + q\delta < 0$, we have, in S_2, that

$$\begin{aligned} J_1(r, t; q) &\leqslant \frac{k}{r} \int_k^{t-r} d\beta \int_{t-r}^{t+r} \left(\frac{\alpha+2k}{k}\right)^{1-q} \left(\log 4 \frac{\alpha+2k}{\beta+2k}\right)^{q} d\alpha \\ &\leqslant \frac{Ck}{r} \int_k^{t-r} \left(\frac{\beta+2k}{k}\right)^{-q\delta} d\beta \int_{t-r}^{t+r} \left(\frac{\alpha+2k}{k}\right)^{1-q+q\delta} d\alpha. \end{aligned} \tag{5.20}$$

Hence the α-integral is estimated as in Case 1 and we obtain

$$J_1(r, t; q) \leqslant Ck^2 \left(\frac{t+r+2k}{k}\right)^{-1} \left(\frac{t-r+2k}{k}\right)^{1-q\delta+2-q+q\delta}. \tag{5.21}$$

Therefore (5.16) follows from (5.4).

Case 5 (Equation (5.1) when $p=q=2$). The desired estimate in this case can be found in F. John [8]. For the sake of completeness, we shall follow his proof. The first inequality in (5.1) follows from

$$I_1(r,t;2)=I_2(r,t;2)\leqslant\frac{k}{r}\int_{-k}^{k}d\beta\int_{t-r}^{t+r}\left(\frac{\alpha+2k}{k}\right)^{-1}d\alpha. \tag{5.22}$$

In order to see the second one, we have to estimate

$$J_1(r,t;2)=J_2(r,t;2)=\frac{k^2}{r}\int_{k}^{t-r}d\beta\int_{t-r}^{t+r}(\alpha-\beta)^{-1}\left(\log\frac{\alpha+2k}{\beta+2k}\right)^2 d\alpha. \tag{5.23}$$

Here we must employ the following inequality.

Lemma 5.1 (F. John [8]).

$$\int_{k}^{t-r}d\beta\int_{t-r}^{t+r}(\alpha-\beta)^{-1}\left(\log\frac{\alpha+2k}{\beta+2k}\right)^2 d\alpha$$
$$\leqslant(t+r+2k)\log\frac{t+r+2k}{t-r+2k}\int_0^1\frac{(\log\theta)^2}{\theta}\,d\theta. \tag{5.24}$$

Proof. See (59d) in [8].

By Lemma 5.1, we get the desired estimate because of $2(2-1)F(2,2)=1$. Now we can conclude that all together the cases complete the proof of Lemma 3.4.

6. UPPER BOUND OF THE LIFESPAN IN THE CRITICAL CASE

Following John's iteration argument together with a new slicing method, we shall estimate the upper bound of the lifespan in the critical case.

Proposition 6.1. *Let $n=3$. Assume that (u,v) is a classical solution of (1.1), (1.2) in the domain $\mathbf{R}^3\times[0,T]$ under the same assumption as that of Theorem 1. Then, for sufficiently small ε and some positive constant C independent of ε, T cannot be taken as*

$$T>\begin{cases}\exp(C\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}) & \text{provided } F(p,q)=0 \text{ with } p\neq q,\\ \exp(C\varepsilon^{-p(p-1)}) & \text{provided } F(p,q)=0 \text{ with } p=q=p_0(3).\end{cases} \tag{6.1}$$

In order to prove the blow-up result, we have to make an iteration frame which can be found in the following lemma. Recall Lemma 2.1.

LEMMA 6.2. *Let* (u, v) *be a classical solution of* (1.1), (1.2) *with the support condition* (2.9). *Assume that* $2 \leqslant p \leqslant q$. *Then there exists a positive constant* M *independent of* ε *such that*

$$\begin{cases} \bar{u}(r, t) \geqslant \dfrac{1}{t+r} \displaystyle\iint_{R(r,t)} \lambda\, |\bar{v}(\lambda, \tau)|^p \, d\lambda\, d\tau + \dfrac{M\varepsilon^p}{(t+r)(t-r)^{p-2}}, \\ \bar{v}(r, t) \geqslant \dfrac{1}{t+r} \displaystyle\iint_{R(r,t)} \lambda\, |\bar{u}(\lambda, \tau)|^q \, d\lambda\, d\tau \end{cases} \quad \text{in } \Sigma_0 \tag{6.2}$$

for sufficiently small ε, *where*

$$\begin{aligned} \Sigma_0 &= \{(r, t) \colon k \leqslant t - r \leqslant r\}, \\ R(r, t) &= \{(\lambda, \tau) \colon t - r \leqslant \lambda,\ \tau + \lambda \leqslant t + r,\ k \leqslant \tau - \lambda \leqslant t - r\}. \end{aligned} \tag{6.3}$$

Remark 6.3. If we assume that the initial data are positive in some sense, Lemma 6.2 becomes an easy application of the single case from H. Takamura [18]. For example, one may assume that

$$f_2(x) \equiv 0 \qquad \text{and} \qquad g_2(x) \geqslant 0 \ (\not\equiv 0) \tag{6.4}$$

while f_1 and g_1 can be arbitrary. In order to remove the positivity on the initial data, we have to use the local existence of solutions which gives us a restriction $p, q \geqslant 2$. As a consequence, once we get a local solution of the associated integral equation (2.1), Lemma 6.2 will be valid for $p, q > 1$ without the positivity on the initial data.

Proof of Lemma 6.2. First, we may assume that $f_2(0) \neq 0$ by a possible shift of the origin. Then, by continuity of v^0, we have $v^0(x, t) \neq 0$ near $(x, t) = (0, 0)$. Taking into account of the sign of v^0 near $(0, 0)$, we also get

$$\overline{v^0}(r, t) \neq 0 \qquad \text{near} \quad (r, t) = (0, 0). \tag{6.5}$$

Next, we shall use the expression of $\overline{v^0}$. Taking the spherical mean in (2.3), we obtain

$$2r\overline{v^0}(r, t) = V^0(t + r) - V^0(t - r), \tag{6.6}$$

where

$$V^0(s) = s\overline{f_2}(|s|) - \int_{|s|}^{\infty} \lambda \overline{g_2}(\lambda)\, d\lambda. \tag{6.7}$$

We note that supp $V^0 \subset [-k, k]$ by assumption on the compactness of the support of the initial data. Then it follows from (6.5) that

$$V^0(s) \neq 0 \qquad \text{near} \quad s=0 \tag{6.8}$$

which implies there exists $s_0 \in [-k, k]$ such that $\eta = |V^0(s_0)| > 0$. Hence there exist k_1, k_2 satisfying $-k < k_1 < k_2 < k$ such that

$$|V^0(s)| \geqslant \frac{\eta}{2} > 0 \qquad \text{for any} \quad s \in [k_1, k_2]. \tag{6.9}$$

The expression of $\overline{v^0}$ also implies that $2r\overline{v^0}(r, t) = -V^0(t-r)$ when $t \geqslant k$. Hereafter, we may assume that $t \geqslant k$ without loss of generality. Therefore we obtain

$$|\overline{v^0}(r, t)| \geqslant \frac{\eta}{4r} \qquad \text{in} \quad S_1' = \{k_1 \leqslant t-r \leqslant k_2\} \subset S_1. \tag{6.10}$$

Now, we shall use the local existence. Hereafter a constant C may change from line to line until the end of the proof of Lemma 6.2. By virtue of Lemma 3.3, we have, in S_1,

$$\tilde{u}(r, t) \leqslant C\varepsilon \begin{cases} \dfrac{k}{t+r+2k} \log 4 \dfrac{t+r+2k}{k} & \text{if} \quad p=2<q, \\ \dfrac{k}{t+r+2k} & \text{otherwise}, \end{cases} \tag{6.11}$$

where

$$\tilde{u}(r, t) = \sup_{|\omega|=1} |u(r\omega, t)|. \tag{6.12}$$

In view of (2.1), we get, in S_1,

$$|v(x, t) - \varepsilon v^0(x, t)| \leqslant \frac{1}{8r} \int_{-k}^{k} d\beta \int_0^{t+r} (\alpha+k)\, \tilde{u}^q \left(\frac{\alpha-\beta}{2}, \frac{\alpha+\beta}{2} \right) d\alpha. \tag{6.13}$$

Along with the proof of Lemma 3.3, one can find

$$|v(x, t) - \varepsilon v^0(x, t)| \leqslant \frac{C\varepsilon}{r} \times \begin{cases} \varepsilon \log \dfrac{2T+3k}{k} & \text{if} \quad p=q=2, \\ \varepsilon^{q-1} & \text{otherwise}. \end{cases} \tag{6.14}$$

When $p=q=2$, due to the local existence, we can take ε so small that

$$\varepsilon \log \frac{2T+3k}{k} \leqslant \delta \tag{6.15}$$

holds for a fixed constant $\delta>0$ as far as a solution exists. Therefore, by (6.10), we can conclude that there exists a positive constant C independent of ε such that $\bar{v}$ satisfies

$$|\bar{v}(r,t)| \geqslant \frac{C\varepsilon}{r} \qquad \text{in } S_1' \tag{6.16}$$

for sufficiently small ε.

Now, we shall follow the argument in H. Takamura [18]. It follows from Lemma 2.3 and the same estimates as [18] that

$$\bar{u}(r,t) \geqslant \frac{1}{2r} \int_{R(r,t)} \lambda\, |\bar{v}(\lambda,\tau)|^p \, d\lambda\, d\tau + H(r,t) \qquad \text{in } \Sigma_0, \tag{6.17}$$

where H is defined by

$$H(r,t) = \frac{1}{2r} \int_{S_1'} \lambda\, |\bar{v}(\lambda,\tau)|^p \, d\lambda\, d\tau. \tag{6.18}$$

Hence, by (6.16) and $t+r \geqslant 3(t-r)$ in Σ_0, we have

$$\begin{aligned} H(r,t) &\geqslant \frac{C^p \varepsilon^p}{2r} \int_{k_1}^{k_2} d\beta \int_{2(t-r)+\beta}^{3(t-r)} \alpha^{1-p}\, d\alpha \\ &\geqslant \frac{C^p \varepsilon^p}{3^{p-1} r (t-r)^{p-1}} \int_{k_1}^{k_2} (t-r-\beta)\, d\beta. \end{aligned} \tag{6.19}$$

When $k_2>0$, it is possible to find a constant $a \in (\max\{k_1 k_2^{-1}, 0\}, 1)$. Therefore we get, in Σ_0,

$$\begin{aligned} H(r,t) &\geqslant \frac{C^p \varepsilon^p}{3^{p-1} r (t-r)^{p-1}} \int_{k_1}^{ak_2} (t-r-\beta)\, d\beta \\ &\geqslant \frac{C^p (1-a)(ak_2-k_1)}{3^{p-1}} \cdot \frac{\varepsilon^p}{r(t-r)^{p-2}}. \end{aligned} \tag{6.20}$$

Taking

$$M = \begin{cases} 3^{1-p} C^p (1-a)(ak_2-k_1) & \text{when } \ k_2>0, \\ 3^{1-p} C^p (k_2-k_1) & \text{when } \ k_2 \leqslant 0, \end{cases} \tag{6.21}$$

we can end the proof of Lemma 6.2.

Proof of Proposition 6.1. Throughout this section we assume that $2 \leqslant p \leqslant q$. In this case, we note that $p \leqslant p_0(3) = 1 + \sqrt{2} \leqslant q$ for

$$F(p, q) \equiv \frac{q + 2 + p^{-1}}{pq - 1} - 1 = 0. \tag{6.22}$$

The opposite case is proved by replacing u, p with v, q, respectively.

Let (u, v) be a classical solution of (1.1), (1.2) in $\mathbf{R}^3 \times [0, T]$. Let us define the blow-up domain. For $j \geqslant 1$,

$$\Sigma_j = \{(r, t) \in \mathbf{R}^2_+ \times [0, T] : l_j k \leqslant t - r \leqslant r\}, \tag{6.23}$$

where $l_j = 1 + 2^{-1} + \cdots + 2^{-j}$. We will use the fact that a sequence $\{l_j\}$ is monotonously increasing and bounded, $1 < l_j < 2$, so $\Sigma_{j+1} \subset \Sigma_j$. This is the *slicing* of the blow-up set.

Assume an estimate of the form

$$\bar{u}(r, t) \geqslant \frac{C_j}{(t + r)(t - r)^{p-2}} \left(\log \frac{t - r}{l_{2j} k} \right)^{a_j} \qquad \text{in } \Sigma_{2j}, \tag{6.24}$$

where $a_j \geqslant 0$ and $C_j > 0$. Inserting (6.24) into the second inequality of (6.2) and noting that $t + r \geqslant 3(t - r)$, we get an estimate of $\bar{v}$ in Σ_{2j+1} such as

$$\begin{aligned} \bar{v}(r, t) &\geqslant \frac{C_j^q}{4(t + r)} \int_{l_{2j} k}^{t - r} \beta^{-q(p-2)} \left(\log \frac{\beta}{l_{2j} k} \right)^{q a_j} d\beta \int_{2(t - r) + \beta}^{3(t - r)} (\alpha - \beta)\, \alpha^{-q}\, d\alpha \\ &\geqslant \frac{C_j^q}{2 \cdot 3^q (t + r)(t - r)^{q-1}} \int_{l_{2j} k}^{t - r} \beta^{-q(p-2)} (t - r - \beta) \left(\log \frac{\beta}{l_{2j} k} \right)^{q a_j} d\beta. \end{aligned} \tag{6.25}$$

At this stage, the proof must be divided into two cases.

Case $p \neq q$. The case follows, by (5.17), from

$$1 - q(p - 2) > 0 \qquad \text{for} \quad F(p, q) = 0 \quad \text{with } p \neq q \tag{6.26}$$

and

$$1-\frac{l_{2j}}{l_{2j+1}}>\frac{1}{2^{2j+2}} \tag{6.27}$$

that the β-integral is greater than

$$\begin{aligned}
&\int_{(l_{2j}/l_{2j+1})(t-r)}^{t-r} \beta^{1-q(p-2)-1}(t-r-\beta)\left(\log\frac{\beta}{l_{2j}k}\right)^{qa_j} d\beta\\
&\quad\geqslant\left(\frac{l_{2j}}{l_{2j+1}}\right)^{1-q(p-2)}(t-r)^{-q(p-2)}\left(\log\frac{t-r}{l_{2j+1}k}\right)^{qa_j}\\
&\qquad\times\int_{(l_{2j}/l_{2j+1})(t-r)}^{t-r}(t-r-\beta)\,d\beta\\
&\quad\geqslant 2^{q(p-2)-2}(t-r)^{-q(p-2)}\left(\log\frac{t-r}{l_{2j+1}k}\right)^{qa_j}\left(1-\frac{l_{2j}}{l_{2j+1}}\right)^2(t-r)^2\\
&\quad\geqslant 2^{q(p-2)-6}\,\frac{1}{16^j}\,(t-r)^{2-q(p-2)}\left(\log\frac{t-r}{l_{2+1}k}\right)^{qa_j}.
\end{aligned} \tag{6.28}$$

Hence we obtain an estimate for $\bar{v}$ such that

$$\bar{v}(r,t)\geqslant\frac{D_j}{(t+r)(t-r)^{pq-q-3}}\left(\log\frac{t-r}{l_{2j+k}k}\right)^{qa\mu_j} \quad \text{in} \quad \Sigma_{2j+1}, \tag{6.29}$$

where we put

$$D_j=3^{-q}2^{q(p-2)-7}\,\frac{C_j^q}{16^j}. \tag{6.30}$$

Similarly, by inserting (6.29) into the first inequality of Lemma 6.2, we have a new estimate for $\bar{u}$ in Σ_{2j+2} as

$$\begin{aligned}
\bar{u}(r,t)&\geqslant\frac{D_j^p}{4(t+r)}\int_{l_{2j+1}k}^{t-r}\frac{\left(\log\dfrac{\beta}{l_{2j+1}k}\right)^{pqa_j}}{\beta^{p(pq-q-3)}}\,d\beta\int_{2(t-r)+\beta}^{3(t-r)}(\alpha-\beta)\,\alpha^{-p}\,d\alpha\\
&\leqslant\frac{D_j^p}{2\cdot 3^p(t+r)(t-r)^{p-1}}\int_{l_{2j+1}k}^{t-r}\frac{t-r-\beta}{\beta}\left(\log\frac{\beta}{l_{2j+1}k}\right)^{pqa_j}d\beta
\end{aligned} \tag{6.31}$$

because

$$p(pq-q-3)=1 \qquad \text{when} \quad F(p,q)=0. \tag{6.32}$$

The integration by parts yields that the β-integral is equal to

$$\frac{1}{pqa_j+1}\int_{l_{2j+1}k}^{t-r}\left(\log\frac{\beta}{l_{2j+1}k}\right)^{pqa_j+1}d\beta. \tag{6.33}$$

Hence, for $(r,t)\in\Sigma_{2j+2}$, the β-integral is greater than

$$\begin{aligned}
&\frac{1}{pqa_j+1}\int_{(l_{2j+1}/l_{2j+2})(t-r)}^{t-r}\left(\log\frac{\beta}{l_{2j+1}k}\right)^{pqa_j+1}d\beta\\
&\quad\leqslant\frac{1}{pqa_j+1}\left(\log\frac{t-r}{l_{2j+2}k}\right)^{pqa_j+1}\left(1-\frac{l_{2j+1}}{l_{2j+2}}\right)(t-r)\\
&\quad\leqslant\frac{1}{2^2 4^j(pqa_j+1)}\left(\log\frac{t-r}{l_{2j+2}k}\right)^{pqa_j+1}(t-r).
\end{aligned} \tag{6.34}$$

Therefore we finally obtain

$$\bar{u}(r,t)\geqslant\frac{C_{j+1}}{(t+r)(t-r)^{p-2}}\left(\log\frac{t-r}{l_{2j+2}k}\right)^{pqa_j+1}\qquad\text{in }\Sigma_{2j+2}. \tag{6.35}$$

where we put

$$C_{j+1}=2^{-3}3^{-p}\frac{D_j^p}{4^j(pqa_j+1)}. \tag{6.36}$$

Now, we are in a position to define sequences in the iteration. In view of (6.2), the original estimate is

$$\bar{u}(r,t)\geqslant\frac{M\varepsilon^p}{(t+r)(t-r)^{p-2}}\qquad\text{in }\Sigma_0 \tag{6.37}$$

so that, with the help of (6.24), (6.29), and (6.35), a sequence $\{a_j\}$ must be defined by

$$\begin{aligned}
a_{j+1}&=pqa_j+1,\qquad j\geqslant 1,\\
a_0&=0.
\end{aligned} \tag{6.38}$$

Another sequence $\{C_j\}$ is determined by

$$\begin{aligned}
C_{j+1}&=2^{-3}3^{-p}\frac{D_j^p}{4^j(pqa_j+1)},\qquad j\geqslant 1,\\
D_j&=3^{-q}2^{q(p-2)-7}\frac{C_j^q}{16^j},\qquad j\geqslant 1,\\
C_0&=M\varepsilon^p.
\end{aligned} \tag{6.39}$$

One can readily check that

$$a_j = \frac{1}{pq-1}\{(pq)^j - 1\}, \qquad j \geqslant 1, \tag{6.40}$$

which gives

$$\frac{1}{pqa_j+1} \geqslant \frac{pq-1}{pq}(pq)^{-j}. \tag{6.41}$$

Hence one can find that

$$C_{j+1} \geqslant E\,\frac{C_j^{pq}}{F^j}, \qquad j \geqslant 1, \tag{6.42}$$

where E and F are positive constants defined by

$$E = \frac{2^{pq(p-2)-7p-3}(pq-1)}{3^{p(q+1)}pq}, \qquad F = 4^{2p+1}pq. \tag{6.43}$$

Repeating this inequality j times, we get

$$\log C_j \geqslant (pq)^j \left(\log C_0 + \sum_{m=1}^{j} \frac{(pq)^{m-1}\log E - (m-1)(pq)^{j-m}\log F}{(pq)^j}\right). \tag{6.44}$$

The sum part of the above inequality converges as $j \to \infty$ by d'Alembert's criterion. It follows that there exists a constant S independent of j such that

$$C_j \geqslant \exp\{(pq)^j(\log C_0 + S)\}, \qquad j \geqslant 1. \tag{6.45}$$

Combining all of the estimates and using the monotonicity of Σ_j, we can reach the final inequality

$$\begin{aligned}\bar{u}(r,t) &\geqslant \frac{C_j}{(t+r)(t-r)^{p-2}}\left(\log\frac{t-r}{2k}\right)^{a_j} \\ &\geqslant \frac{\exp\{(pq)^j I(r,t)\}}{(t+r)(t-r)^{p-2}}\left(\log\frac{t-r}{2k}\right)^{-(pq-1)^{-1}}\end{aligned} \tag{6.46}$$

for $(r,t) \in \Sigma_\infty = \{(r,t) \in \mathbf{R}_+ \times [0,T) : 2k < t-r \leqslant r\}$, where we put

$$I(r,t) = \log\left\{Me^S\varepsilon^p\left(\log\frac{t-r}{2k}\right)^{(pq-1)^{-1}}\right\}. \tag{6.47}$$

At this stage, it is clear that there exists a point $(t_0/2, t_0) \in \Sigma_\infty$ such that $I(t_0/2, t_0) > 0$ provided

$$T > 4k \exp\{(Me^S)^{1-pq} \varepsilon^{-p(pq-1)}\}. \tag{6.48}$$

Taking $j \to \infty$, we get a desired result $\bar{u}(t_0/2, t_0) \to \infty$ which contradicts the assumption that u is a classical solution in $\mathbf{R}^3 \times [0, T]$. So, the proof of the critical case for $p \neq q$ is complete.

Case $p = q = p_0(3)$. This case will be proved in almost the same way as a previous case. In fact, nothing changes up to the inequality (6.25). From now on, we denote q by p.

It follows from the definition of $p_0(3) = 1 + \sqrt{2}$ and the related quadratic equation

$$\gamma(p, 3) \equiv 2 + 4p - 2p^2 = 0 \qquad \text{with} \quad p = p_0(3) \tag{6.49}$$

that

$$1 - p(p-2) = 0 \qquad \text{when} \quad F(p, p) = 0 \quad \text{with } p = p_0(3). \tag{6.50}$$

Therefore, the β-integral in (6.25) is equal to

$$\int_{l_{2j}}^{t-r} \frac{t-r-\beta}{\beta} \left(\log \frac{\beta}{l_{2j}k}\right)^{pa_j} d\beta. \tag{6.51}$$

So, the integration by parts yields that the β-integral must be

$$\frac{1}{pa_j + 1} \int_{l_{2j}}^{t-r} \left(\log \frac{\beta}{l_{2j}k}\right)^{pa_j+1} d\beta. \tag{6.52}$$

Hence, by slicing again, we obtain the estimate for $\bar{v}$,

$$\bar{v}(r, t) \geqslant \frac{D'_j}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{l_{2j+1}k}\right)^{pa_j+1} \qquad \text{in } \Sigma_{2j+1}, \tag{6.53}$$

where we put

$$D'_j = 2^{-3} 3^{-p} \frac{C_j^p}{4^j (pa_j + 1)}, \tag{6.54}$$

instead of (6.29).

Similar to the previous case, we have a new estimate for $\bar{u}$ in Σ_{2j+2} as follows:

$$\begin{aligned}\bar{u}(r,t) &\geqslant \frac{(D_j')^p}{4(t+r)} \int_{l_{2j+1}k}^{t-r} \frac{\left(\log \dfrac{\beta}{l_{2j+1}k}\right)^{p(pa_j+1)}}{\beta^{p(p-2)}}\, d\beta \int_{2(t-r)+\beta}^{3(t-r)} (\alpha-\beta)\,\alpha^{-p}\, d\alpha \\ &\geqslant \frac{(D_j')^p}{2\cdot 3^p(t+r)(t-r)^{p-1}} \int_{l_{2j+1}k}^{t-r} \frac{t-r-\beta}{\beta} \left(\log \frac{\beta}{l_{2j+1}k}\right)^{p(pa_j+1)} d\beta.\end{aligned} \tag{6.55}$$

Hence the same treatment on the β-integral implies a new estimate

$$\bar{u}(r,t) \geqslant \frac{C_{j+1}}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{l_{2j+2}k}\right)^{p(pa_j+1)+1} \quad \text{in } \Sigma_{2j+2}, \tag{6.56}$$

where we put

$$C_{j+1} = 2^{-3}3^{-p} \frac{(D_j')^p}{4^j(p(pa_j+1)+1)}. \tag{6.57}$$

Now, we can define sequences as before. $\{a_j\}$ is defined by

$$\begin{aligned} &a_{j+1} = p(pa_j+1)+1, \qquad j \geqslant 1, \\ &a_0 = 0 \end{aligned} \tag{6.58}$$

and $\{C_j\}$ is determined by

$$\begin{cases} C_{j+1} = 2^{-3}3^{-p} \dfrac{(D_j')^p}{4^j(p(pa_j+1)+1)}, & j \geqslant 1, \\ D_j' = 2^{-3}3^{-p} \dfrac{C_j^p}{4^j(pa_j+1)}, & j \geqslant 1, \\ C_0 = M\varepsilon^p. \end{cases} \tag{6.59}$$

One can readily check that

$$a_j = \frac{1}{p-1}(p^{2j}-1), \quad j \geqslant 1, \tag{6.60}$$

which gives

$$\frac{1}{p(pa_j+1)+1} \geqslant \frac{p-1}{p^2}\, p^{-2j}, \qquad \frac{1}{pa_j+1} \geqslant \frac{p-1}{p}\, p^{-2j}. \tag{6.61}$$

Hence one can find that

$$C_{j+1} \geqslant E \frac{C_j^{p^2}}{F^j}, \qquad j \geqslant 1, \tag{6.62}$$

where E and F are positive constants defined by

$$E = \frac{1}{(2^3 3^p)^{p+1} p} \left(\frac{p-1}{p} \right)^{p+1}, \qquad F = (2p)^{2(p+1)}. \tag{6.63}$$

This is the same form as in the previous case. So, the same reason shows that, after repeating this inequality j-times, one can find the existence of a constant S independent of j such that

$$C_j \geqslant \exp\{ p^{2j} (\log C_0 + S) \}, \qquad j \geqslant 1. \tag{6.64}$$

Combining all of the estimates, we can reach the final inequality

$$\bar{u}(r, t) \geqslant \frac{\exp\{ p^{2j} I'(r, t) \}}{(t+r)(t-r)^{p-2}} \left(\log \frac{t-r}{2k} \right)^{-(p-1)^{-1}} \tag{6.65}$$

for $(r, t) \in \Sigma_\infty = \{ (r, t) \in \mathbf{R}_+ \times [0, T] : 2k < t - r \leqslant r \}$, where we put

$$I'(r, t) = \log \left\{ M e^S \varepsilon^p \left(\log \frac{t-r}{2k} \right)^{(p-1)^{-1}} \right\}. \tag{6.66}$$

Hence there exists a point $(t_0/2, t_0) \in \Sigma_\infty$ such that $I(t_0/2, t_0) > 0$ provided

$$T > 4k \exp\{ (M e^S)^{1-p} \varepsilon^{-p(p-1)} \}. \tag{6.67}$$

Taking $j \to \infty$, we get a desired contradiction. The proof is now complete.

Remark 6.4. We note that the above proof never requires that $p \geqslant 2$ except for the assumption in Lemma 6.2. This means that Proposition 6.1 may still be valid for C^1-solutions of associated integral equations (2.1) with a low power $1 < p < 2$. See Remark 6.3.

7. UPPER BOUND OF THE LIFESPAN IN THE SUB-CRITICAL CASE

In the sub-critical case, we can estimate the upper bound of the lifespan only by John's iteration argument. There is no need to use slicing.

PROPOSITION 7.1. *Let $n = 3$. Assume that (u, v) is a classical solution of* (1.1), (1.2) *in the domain $\mathbf{R}^3 \times [0, T]$ under the same assumption as that* *in Theorem* 1. *Then, for sufficiently small* ε *and some positive constant* C *independent of* ε, T *cannot be taken as*

$$T > C\varepsilon^{-F(p,q)^{-1}} \qquad \textit{provided} \quad F(p,q) > 0. \tag{7.1}$$

Proof of Proposition 7.1. We shall follow the proof of Proposition 6.1 without the slicing. It is sufficient to consider the case $2 \leqslant p \leqslant q$ as before. Let (u, v) be a classical solution of (1.1), (1.2) in $\mathbf{R}^3 \times [0, T]$.

Instead of (6.24), assume an estimate of the form

$$\bar{u}(r,t) \geqslant \frac{C_j (t-r-k)^{a_j}}{(t+r)(t-r)^{p-2+b_j}} \qquad \text{in } \Sigma_0, \tag{7.2}$$

where $a_j, b_j \geqslant 0$ and $C_j > 0$. Putting (7.2) into the second inequality of (6.2), we get an estimate of $\bar{v}$ in Σ_0 such as

$$\begin{aligned} \bar{v}(r,t) &\geqslant \frac{C_j^q}{4(t+r)} \int_k^{t-r} \beta^{-q(p-2+b_j)} (\beta-k)^{qa_j}\, d\beta \int_{2(t-r)+\beta}^{3(t-r)} (\alpha-\beta)\, \alpha^{-q}\, d\alpha \\ &\geqslant \frac{C_j^q}{2\cdot 3^q (t+r)(t-r)^{q-1+qb_j}} \int_k^{t-r} \beta^{-q(p-2)} (\beta-k)^{qa_j}\, (t-r-\beta)\, d\beta. \end{aligned} \tag{7.3}$$

It follows from $-q(p-2) \leqslant 0$ that

$$\bar{v}(r,t) \geqslant \frac{C_j^q}{2\cdot 3^q (t+r)(t-r)^{q-1+q(p-2+b_j)}} \int_k^{t-r} (\beta-k)^{qa_j}\, (t-r-\beta)\, d\beta. \tag{7.4}$$

The integration by parts yields that the β-integral is equal to

$$\int_k^{t-r} \frac{(\beta-k)^{qa_j+1}}{qa_j+1}\, d\beta \geqslant \frac{(t-r-k)^{qa_j+2}}{(qa_j+2)^2}. \tag{7.5}$$

Hence we obtain an estimate for $\bar{v}$ such that

$$\bar{v}(r,t) \geqslant \frac{D_j (t-r-k)^{qa_j+2}}{(t+r)(t-r)^{q(p-2+b_j)+q-1}} \qquad \text{in } \Sigma_0, \tag{7.6}$$

where

$$D_j = \frac{C_j^q}{2\cdot 3^q (qa_j+2)^2}. \tag{7.7}$$

Similarly, putting (7.6) into the first inequality of Lemma 6.2, we have a new estimate for $\bar{u}$ in Σ_0 as follows.

$$\bar{u}(r,t) \geqslant \frac{D_j^p}{2\cdot 3^p(t+r)(t-r)^{p-1}}$$

$$\times \int_k^{t-r} \beta^{-p(q(p-2+b_j)+q-1)}(\beta-k)^{p(qa_j+2)}(t-r-\beta)\,d\beta$$

$$\geqslant \frac{D_j^p}{2\cdot 3^p(t+r)(t-r)^{p-1+p(q(p-2+b_j)+q-1)}}$$

$$\times \int_k^{t-r} (\beta-k)^{p(qa_j+2)}\,(t-r-\beta)\,d\beta. \tag{7.8}$$

The integration by parts yields that the β-integral is greater than

$$\frac{(t-r-k)^{p(qa_j+2)+2}}{\{p(qa_j+2)+2\}^2}. \tag{7.9}$$

Therefore we finally obtain

$$\bar{u}(r,t) \geqslant \frac{C_{j+1}(t-r-k)^{p(qa_j+2)+2}}{(t+r)(t-r)^{p-2+p(q(p-2+b_j)+q-1)+1}} \qquad \text{in } \Sigma_0, \tag{7.10}$$

where

$$C_{j+1} \geqslant \frac{D_j^p}{2\cdot 3^p\{p(qa_j+2)+2\}^2}. \tag{7.11}$$

Now, we are in a position to define the sequences in the iteration. In view of (6.2), the original estimate is

$$\bar{u}(r,t) \geqslant \frac{M\varepsilon^p}{(t+r)(t-r)^{p-2}} \qquad \text{in } \Sigma_0 \tag{7.12}$$

so that, with the help of (7.2), (7.6), and (7.10), sequences $\{a_j\}$ and $\{b_j\}$ must be defined by

$$\begin{aligned} a_{j+1} &= pqa_j + 2p + 2, \qquad j \geqslant 1, \\ a_0 &= 0, \end{aligned} \tag{7.13}$$

and

$$\begin{aligned} b_{j+1} &= pqb_j + (p-1)(pq-1), \qquad j \geqslant 1, \\ b_0 &= 0. \end{aligned} \tag{7.14}$$

Another sequence $\{C_j\}$ is determined by

$$\begin{aligned} C_{j+1} &= \frac{D_j^p}{2 \cdot 3^p \{p(qa_j+2)+2\}^2}, \qquad j \geqslant 1, \\ D_j &= \frac{C_j^q}{2 \cdot 3^q (qa_j+2)^2}, \qquad j \geqslant 1, \\ C_0 &= M\varepsilon^p. \end{aligned} \tag{7.15}$$

One can readily check that

$$a_j = \frac{2p+2}{pq-1} \{(pq)^j - 1\}, \qquad j \geqslant 1, \tag{7.16}$$

and

$$b_j = (p-1)\{(pq)^j - 1\}, \qquad j \geqslant 1, \tag{7.17}$$

which gives

$$qa_j + 2 \leqslant \frac{2q(p+1)(pq)^j}{pq-1}. \tag{7.18}$$

Hence one can find that

$$C_{j+1} \geqslant E \frac{C_j^{pq}}{F^j}, \qquad j \geqslant 1, \tag{7.19}$$

where E and F are positive constants defined by

$$E = \frac{(pq-1)^{2(p+1)}}{2^{5p+7} 3^{p(q+1)} p^{2(p+2)} q^{2(p+1)}}, \qquad F = (pq)^{2(p+1)}. \tag{7.20}$$

This is the same form as in the critical case. So, the same reasoning shows that, after repeating this inequality j-times, one can find the existence of a constant S independent of j such that

$$C_j \geqslant \exp\{(pq)^j (\log C_0 + S)\}, \qquad j \geqslant 1. \tag{7.21}$$

Combining all estimates, we can reach the final inequality

$$\bar{u}(r,t) \geqslant \frac{(t-r-k)^{-2(p+1)/(pq-1)}}{(t+r)(t-r)^{p-2-(p-1)}} \exp\{(pq)^j I(r,t)\} \qquad \text{in } \Sigma_0, \tag{7.22}$$

where

$$I(r,t) = \log(Me^S \varepsilon^p (t-r-k)^{2(p+1)/(pq-1)} (t-r)^{-(p-1)}). \tag{7.23}$$

It follows from $2(p+1)/(pq-1)-(p-1)=pF(p,q)$ that there exists a point

$$(t_0/2, t_0) \in \{(r,t)\colon 2k \leqslant t-r \leqslant r\} \subset \Sigma_0 \tag{7.24}$$

such that $I(t_0/2, t_0) > 0$ provided

$$T > 2^{1+2(p+1)/p(pq-1)\,F(p,q)} (Me^S)^{-(pF(p,q))^{-1}} \varepsilon^{-F(p,q)^{-1}}. \tag{7.25}$$

Taking $j \to \infty$, we get a desired contradiction.

Remark 7.2. In the case $1<p<2$, we may have the same result as that for the C^1-solution of the associated integral equations (2.1). Actually, instead of (7.4), we get an estimate of $\bar{v}$ such as

$$\bar{v}(r,t) \geqslant \frac{C_j^q}{2\cdot 3^q (t+r)(t-r)^{q-1+qb_j}} \int_k^{t-r} (\beta-k)^{qa_j-q(p-2)} (t-r-\beta)\, d\beta. \tag{7.26}$$

Therefore one can readily check that the same conclusion is still valid by a similar proof because there is no loss of the total power of $t-r-k$ and $t-r$.

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